# AN ALGEBRAIC PROOF OF INVARIANCE FOR KNOT FLOER HOMOLOGY

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Abstract. We present a proof of the invariance of knot Floer homology using the cube of resolutions construction first described in [12]. Specifically, we show that the cube of resolutions chain complex is invariant up to chain homotopy equivalence and base change under the Markov moves. The techniques echo those employed to prove the invariance of HOMFLY-PT homology in [5]. In particular, we make no mention of holomorphic disks or grid diagrams.

## 1. Introduction

Several knot polynomials were originally categorified using a "cube of resolutions" construction. Given a projection of a knot with $m$ crossings, one considers two ways of resolving each crossing and arranges all possible resolutions into an $m$-dimensional cube. To each vertex of the cube, one associates a graded algebraic object (perhaps a vector space, or a module over some commutative ring), and to each edge of the cube a map. With the correct choices of objects and maps, the result is a chain complex whose graded Euler characteristic is the desired knot polynomial. Khovanov's categorification of the Jones polynomial [2] follows this model, employing a cube in which the resolutions are the two possible smoothings of a crossing. A complete resolution is then a collection of circles, to which ones associates certain vector spaces. Khovanov and Rozansky's categorification of the $sl_n$ polynomials [4] and later the HOMFLY-PT polynomial [5] (see also Khovanov [3] and Rasmussen [13]) instead use a cube of resolutions built from singularizations of crossings and oriented smoothings. The complete resolutions in this case are a particular type of oriented planar graph. The associated algebraic objects are modules over the ring $\mathbb{Q}[x_0, \ldots, x_n]$, which has one indeterminate for each edge of the graph. In each of these theories, the chain complex was proved to be a knot invariant by directly checking invariance under Reidemeister moves. That is, one compares the prescribed chain complex before and after a Reidemeister move is performed on the diagram, and constructs a chain homotopy between the two complexes.

Knot Floer homology, which categorifies the Alexander polynomial, was originally developed via an entirely different route. It was defined by Ozsváth and Szabó [10] and by Rasmussen [14] as a filtration on the chain complex of Heegaard Floer homology [11], a three-manifold invariant whose differentials count holomorphic disks in the symmetric product of a surface. Knots in this theory are represented by decorating the Heegaard diagram for a three-manifold, so invariance was proved by checking invariance under Heegaard moves.

In 2007, Ozsváth, Szabó and Stipsicz [9] described a version of knot Floer homology for singular knots that is related to the theory for classical knots by a skein exact

sequence. In general, knot Floer homology for singular knots involves holomorphic disk counts, but it can be made combinatorial with a suitable choice of twisted coefficients and a particular Heegaard diagram. Using this version of the theory for singular knots and iterating the skein exact sequence allowed Ozsváth and Szabó [12] to calculate knot Floer homology from the cube of resolutions perspective.

The goal of this paper is to give a direct proof of the invariance of knot Floer homology within the algebraic setting of the cube of resolutions chain complex, without relying on Heegaard diagrams, holomorphic disks, or any of the usual geometric input. We use a small modification of the cube of resolutions construction described by Ozsváth and Szabó [12] to obtain chain complexes with homology $\widehat{HFK}$ and $HFK^-$.

The construction begins with a projection of a knot as a closed braid, which we decorate with a basepoint and a number of extra bivalent vertices to create a *layered braid diagram* $D$. Each layer of the diagram contains a single crossing and a bivalent vertex on each strand not involved in the crossing. This amounts to choosing a braid word that represents the knot and subdividing some edges. We form a cube of resolutions by singularizing or smoothing each crossing of the projection. We then assign a graded algebra $\mathcal{A}_I(D)$ to each resolution and arrange these into a chain complex $C(D)$. These objects are defined precisely in (1) and (2) of Section 2. Our main result is an algebraic proof of

**Theorem 1.** *Let $D$ be a layered braid diagram for a knot $K$. The chain complex $C(D)$, up to chain homotopy equivalence and base change, is invariant under the Markov moves. Therefore, $H_*(C(D))$ depends only on the knot $K$.*

Theorem 1 holds with coefficients in $\mathbb{Z}$. It is stated in full detail in Section 2.4. Changing to $\mathbb{F}_2$ coefficients, we identify $H_*(C(D))$ with $HFK^-$ and a reduced version of $C(D)$ with $\widehat{HFK}$ in Proposition 9. We expect that $H_*(C(D))$ in fact computes knot Floer homology with integer coefficients, but do not pursue this point here.

The proof of invariance under braid-like Reidemeister moves II and III is very closely modeled on Khovanov and Rozansky's proof for HOMFLY-PT homology in [5]. Specifically, we prove categorified versions of the braid-like MOY relations [7], specialized as appropriate for the Alexander polynomial. That these relations hold in a homology theory that categorifies the Alexander polynomial may be further evidence of a close relationship between knot Floer homology and HOMFLY-PT homology. In fact, implicit in the definitions of $\mathcal{A}_I(D)$ and $C(D)$ is a braid group action similar to that of [3] from which one may recover either knot Floer homology or a variation on the HOMFLY-PT homology of [3] and [5]. We plan to explore the properties of this braid invariant in more detail in a future paper.

There is a third description of $\widehat{HFK}$ developed using grid diagrams in [15] and [8]. This definition is fully combinatorial and its invariance was proved combinatorially in [6]. It was also used to calculate $\widehat{HFK}$ for knots up to 11 crossings in [1]. At this point, the cube of resolutions description has not been used for computations, but it has the potential advantage of a much smaller chain complex. For a knot with arc index $n$, the grid diagrams chain complex has $n!$ generators. The number of generators in the cube of resolutions complex is at most exponential in the minimal braid length. In the case of the trefoil, for instance, the grid diagram complex has 120 generators while the cube of resolutions complex has only 13. This numerical

advantage is tempered by the fact that the cube of resolutions complex requires computations over a larger ground ring, but the cube of resolutions complex may still be a suitable computational tool in some situations.

This paper is organized as follows. Section 2 describes the modified construction of the cube of resolutions needed to incorporate layered braid diagrams. Section 3 examines in detail the non-local relations involved in the definition of the algebra associated to a resolution. These relations are a key difference between the cube of resolutions theories for $HFK$ and for HOMFLY-PT homology. Section 4 establishes a technical proposition allowing us to remove sets of bivalent vertices under certain conditions. The next sections address each of the Markov moves in turn. Section 9 verifies that the cube of resolutions defined here computes knot Floer homology.

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## 2. Definitions: Cube of Resolutions for $HFK$

We begin with an oriented braid-form projection $D$ of an oriented knot $K$ in $S^3$. Let $b$ refer to the number of strands in $D$ (which is not necessarily the braid index of $K$). Subdivide one of the outermost edges of $D$ by a basepoint $*$. Isotoping $D$ as necessary, fix an ordering on its crossings so that $D$ is the closure of a braid diagram that is a stack of $m+1$ horizontal layers, each containing a single crossing and $b-2$ vertical strands. Label the horizontal layers $s_0, \dots, s_m$. This amounts to choosing a braid word for $D$. In each horizontal layer, add a bivalent vertex to each strand that is not part of the crossing. Finally, label the edges of $D$ by $0, \dots, n$ such that 0 is the edge coming out from the basepoint and $n$ is the edge pointing into the basepoint. A braid diagram in this form will be called a *layered braid diagram* for $K$. See Figure 1 for an example of a layered braid diagram of the figure 8 knot. Although Ozsváth and Szabó [12] use closed braid diagrams with basepoints in their definition of the knot Floer cube of resolutions, they do not require diagrams to be layered. This refinement appears to be critical to our proof of Proposition 2 and necessary for the proof of Reidemeister III invariance as well.

Each crossing in a knot projection can be singularized or smoothed. To singularize the crossing in layer $s_i$, replace it by a 4-valent vertex and retain all edge labels. To smooth the crossing in layer $s_i$, replace it with two vertical strands with one bivalent vertex on each, and retain all edge labels. Figure 2 illustrates these labeling conventions.

A resolution of a knot projection is a diagram in which each crossing has been singularized or smoothed. Alternatively, it is a planar graph in which each vertex is either (1) 4-valent with orientations as in Figure 2, or (2) bivalent with one incident edge oriented towards the vertex and the other oriented away from the vertex. For a positive crossing, declare the singularization to be the 0-resolution and the smoothing to be the 1-resolution. For a negative crossing, reverse these labels. A resolution of a knot projection can then be specified by a multi-index of 0s and 1s, generically denoted $\epsilon_0 \dots \epsilon_m$, or simply $I$, which we will think of as a vertex of a hypercube. Considering all possible singularizations and smoothings of

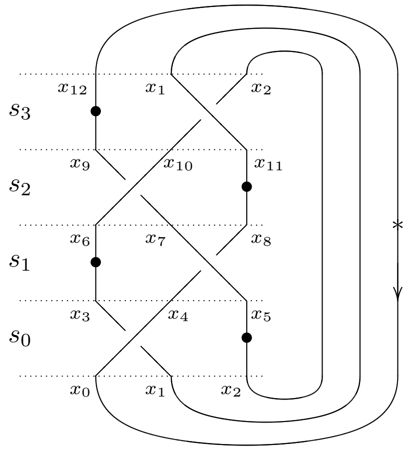



FIGURE 1. A layered braid diagram for the figure 8 knot.

all crossings, we obtain a cube of resolutions for the original knot projection. The homological grading on the cube will be given by collapsing diagonally; that is, by summing $\epsilon_0 + \cdots + \epsilon_m$.

Let $\mathcal{R} = \mathbb{Z}[t^{-1}, t]$ and $\underline{x}(D)$ denote a set of formal variables $x_0, \ldots, x_n$ corresponding to the edges of $D$. Define the *edge ring* of $D$ to be $\mathcal{R}[\underline{x}(D)]$, which we will abbreviate to $\mathcal{R}[\underline{x}]$ if $D$ is clear from context. To each vertex of the cube of resolutions, we will associate an $\mathcal{R}$-algebra $\mathcal{A}_I(D)$, which is a quotient of the edge ring by an ideal defined by combinatorial data in the $I$-resolution of $D$. To each edge of the cube, we will associate a map. Together with proper choices of gradings, these data define a chain complex of graded algebras over $\mathcal{R}[\underline{x}(D)]$. We will sometimes need to complete $\mathcal{R}$ or $\mathcal{R}[\underline{x}(D)]$ with respect to $t$, meaning that we will allow Laurent series in $t$ with coefficients in $\mathbb{Z}$ or $\mathbb{Z}[\underline{x}(D)]$, respectively. Denote these completions $\widehat{\mathcal{R}}$ and $\widehat{\mathcal{R}[\underline{x}(D)]}$, respectively. More specifically, the proof of invariance under stabilization requires extending the base ring to $\widehat{\mathcal{R}}[\underline{x}(D)]$ and the identification of the homology of $C(D)$ knot Floer homology requires extending to $\widehat{\mathcal{R}[\underline{x}(D)]}$ (as well as passing to $\mathbb{F}_2$ coefficients).

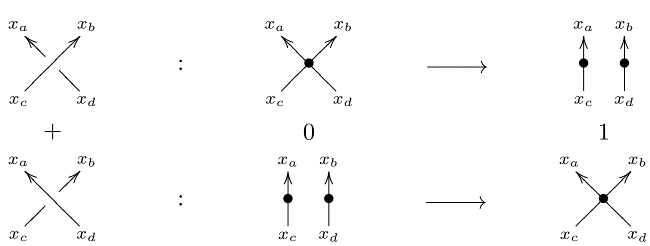
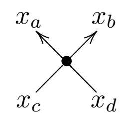



FIGURE 2. Notation for the singularization and smoothing of a positive (respectively negative) crossing.

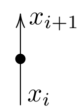
2.1. **Algebra associated to a resolution.** The algebra associated to the $I$-resolution of the knot projection $D$, which we will denote $\mathcal{A}_I(D)$, is the quotient of the edge ring by the ideal generated by the following three types of relations.

(1) Linear relations associated to each vertex.

$$t(x_a + x_b) - (x_c + x_d) \quad \text{to}$$

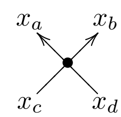



$$tx_{i+1} - x_i \quad \text{to}$$




(2) Quadratic relations associated to each 4-valent vertex.

$$t^2 x_a x_b - x_c x_d \quad \text{to}$$




Note that this relation can always be rewritten in four different ways by combining with the linear relation corresponding to the same vertex:

$$\begin{array}{ccccc} (tx_a - x_c)(x_d - tx_a) & \text{or} & (tx_b - x_c)(x_d - tx_b) & \text{or} \\ (tx_a - x_c)(tx_b - x_c) & \text{or} & (tx_a - x_d)(tx_b - x_d). \end{array}$$

(3) Non-local relations associated to sets of vertices in the resolved diagram. These have several equivalent definitions, which will be explored in detail in Section 3. Denote the ideal generated by non-local relations in $I$-resolution of $D$ by $\mathcal{N}_I(D)$ or simply $\mathcal{N}_I$.

We refer to the linear and quadratic relations as the local relations. Let $\mathcal{L}$ denote the ideal they generate together, and $\mathcal{L}_i$ denote the ideal generated by the local relations in layer $s_i$. Then we have defined the algebras that belong at the corners of the cube of resolutions as

$$\mathcal{A}_I(D) = \frac{\mathcal{R}[x_0, \ldots, x_n]}{\mathcal{L} + \mathcal{N}_I}. \tag{1}$$

Throughout this paper, we will use "$\equiv$" to indicate that two polynomials in the edge ring are equivalent up to multiplication by units in $\mathcal{R}[\underline{x}(D)]/\mathcal{L}$. Such polynomials are equivalent in the sense that they generate the same ideal in $\mathcal{R}[\underline{x}(D)]/\mathcal{L}$. We will represent generating sets for ideals as single-column matrices. The entries of the matrices are elements of the edge ring. The matrices can be manipulated using row operations without changing the ideal they generate because the ideal $(a, b)$ is identical to the ideal $(a, b + sa)$ for any unit $s \in \mathcal{R}$ and $a, b \in \mathcal{R}[\underline{x}(D)]$. Also, when we see a row of the form $a - b$ in a matrix, we can replace $b$ by $a$ in all other rows and eliminate $b$ from the edge ring. This will not change the quotient of the edge ring by the ideal of relations. Although the matrix manipulations in the following sections look very similar to those in [5] and [13], the matrices here do not formally represent matrix factorizations.

The algebra $\mathcal{A}_I(D)$ is a twisted version of the singular knot Floer homology of the $I$-resolution of $D$ treated as a singular knot with bivalent vertices. Reverting to $\mathbb{F}_2$

coefficients and setting $x_a = x_b$ at each 4-valent vertex and $x_0$ to zero, then taking homology, gives the theory called $HFS$ in [9]. Setting all of the edge variables to zero before taking homology gives the theory called $\widetilde{HFS}$ in [9]. Both $HFS$ and $\widetilde{HFS}$ categorify the singular Alexander polynomial (or a multiple thereof). However, $HFS$ is completely determined by the singular Alexander polynomial, while $\widetilde{HFS}$ contains additional information.

2.2. **Differential.** An edge of the cube of resolutions goes between two resolutions that differ at exactly one crossing. To an edge that changes the $i^{th}$ crossing, we associate a map $\mathcal{A}_{\epsilon_0\ldots 0\ldots\epsilon_m}(D) \longrightarrow \mathcal{A}_{\epsilon_0\ldots 1\ldots\epsilon_m}(D)$. If $s_i$ was positive in the original knot projection, then the edge goes from a diagram containing the singularization of $s_i$ to a diagram containing its smoothing. The ideal of relations associated to the singularized crossing is contained in the ideal of relations associated to the resolved crossing, so $\mathcal{A}_{\epsilon_0\ldots 1\ldots\epsilon_m}(D)$ is a quotient of $\mathcal{A}_{\epsilon_0\ldots 0\ldots\epsilon_m}(D)$. The corresponding map in this case will be the quotient map. If $s_i$ was negative in the original knot projection, then the edge goes from the smoothing to the singularization of $s_i$. The corresponding map in this case will be multiplication by $tx_a - x_d$, or equivalently by $tx_b - x_c$, where the crossing $s_i$ is labeled as in Figure 2.

We have now assembled all of the pieces needed to define the chain complex $(C(D), d)$ referred to in Theorem 1. Let

$$C(D) = \bigoplus_{I\in\{0,1\}^{m+1}} \mathcal{A}_I(D) \tag{2}$$

with total differential $d$ the sum of all edge maps and homological grading given by $\epsilon_0 + \cdots + \epsilon_m$. This is the chain complex that computes $HFK^-$ (see Proposition 9). There is also a reduced version of this chain complex obtained by setting $x_0$ to zero in each $\mathcal{A}_I(D)$. Its homology computes $\widehat{HFK}$.

2.3. **Gradings.** The chain complex $C(D)$ comes equipped with an additional grading called the Alexander grading. Let $\mathcal{R}$ be in grading 0 and each edge variable $x_i$ in grading -1. The relations used to form $\mathcal{A}_I(D)$ are homogeneous with respect to this grading, so it descends from the edge ring to a grading on $\mathcal{A}_I(D)$ (called $A_0$ in [12]). To symmetrize, adjust upwards by a factor of $\frac{1}{2}(\sigma - b + 1)$, where $\sigma$ is the number of singular points in the $I$-resolution of $D$ and $b$ is the number of strands in $D$. Call this the internal grading, $A_I$, on $\mathcal{A}_I(D)$.

The Alexander grading on $\mathcal{A}_I(D)$ as a summand of the cube $C(D)$ is further adjusted from the internal grading by

$$A = A_I + \frac{1}{2}\left(-N + \sum_{i=0}^{m} \epsilon_i\right),$$

where $\epsilon_0, \ldots, \epsilon_m$ are the components of the multi-index $I$ and $N$ is the number of negative crossings in $D$. This grading $A$ is the final Alexander grading on the complex $C(D)$.

2.4. **Invariance.** With these definitions in place, we may now state Theorem 1 precisely.

**Theorem.** *Let $D$ be a layered braid diagram with initial edge $x_0$ representing a knot $K$ in $S^3$. As a complex of graded $\widehat{\mathcal{R}}[x_0]$-modules up to chain homotopy equivalence*

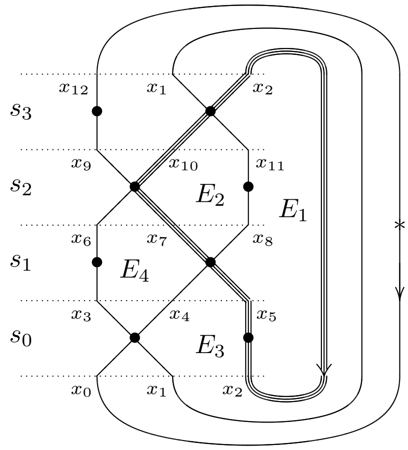



FIGURE 3. Singularization of the minimal braid presentation of the figure 8 knot with edges labeled $x_0, \ldots, x_{12}$ and orientations consistent with those in Figure 1. The bold line shows a cycle whose corresponding non-local relation is $t^8 x_1 x_9 - x_4 x_6$. Elementary regions are labeled $E_1, \ldots, E_4$. The coherent region $E_1 \cup E_2$ produces the same non-local relation as the cycle in bold, as does the subset consisting of the bivalent vertex in $s_0$, the 4-valent vertex in $s_1$, all vertices in $s_2$, and the 4-valent vertex in $s_3$.

*and base change, $C(D) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$ is invariant under Markov moves on $D$. Therefore, $H_*(C(D)) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$ is an invariant of the knot $K$.*

Note that the last statement relies on the flatness of $\widehat{\mathcal{R}}$ as a $\mathcal{R}$-module.

## 3. Non-local Relations

We collect here three equivalent definitions of the non-local relations used in the description of the algebra $\mathcal{A}_I(D)$, along with several straightforward observations that will nonetheless be very useful in later arguments. Figure 3 will serve as a source of examples throughout.

First, we may generate $\mathcal{N}_I$ by associating a relation to each cycle (closed path) in the resolved diagram that does not pass through the basepoint and that is oriented consistently with $D$.

**Definition 1** (Cycles)**.** *Let $Z$ be a closed path in the $I$-resolution of $D$ that does not pass through the basepoint and is oriented consistently with $D$. Let $R_Z$ be the region it bounds in the plane, containing the braid axis. The* weight $\mathbf{w}(Z)$ *of $Z$ is twice the number of 4-valent vertices plus the number of bivalent vertices in the closure of $R_Z$. The* non-local relation associated to $Z$ *is*

$$t^{\mathbf{w}(Z)} w_{out} - w_{in},$$

*where $w_{out}$ (respectively $w_{in}$) is the product of the edges incident to exactly one vertex of $Z$ that lie outside of $R_Z$ and that point out of (respectively into) the region.*

Figure 3 shows a cycle in the singularized figure 8 knot with associated relation $t^8x_1x_9 - x_4x_6$.

A slightly different definition derives a generating set for $\mathcal{N}_I$ from certain regions in the complement of the $I$-resolution of $D$. First define the *elementary regions* in the $I$-resolution of $D$ to be the connected components of its complement in the plane, except for the two components that are adjacent to the basepoint. For example, there are four elementary regions in the singularized figure 8 shown in Figure 3.

Since $D$ is assumed to be in braid position, the elementary regions can be partially ordered based on which two strands of $D$ they lie between. Label the strands of $D$ from 1 (innermost, nearest the braid axis) to $b$ (outermost, containing the basepoint). Then $E_i < E_j$ with respect to the partial order if $E_i$ is closer to the braid axis than $E_j$; that is, if $E_i$ lies between lower-numbered strands than $E_j$ does. Let $E_1$ denote the innermost elementary region, containing the braid axis. Label the other elementary regions $E_2, \ldots, E_m$ so that whenever $i < j$, $E_i$ is less than or not comparable to $E_j$ with respect to the partial order.

**Definition 2** (Coherent Regions)**.** *A* coherent region *in the $I$-resolution of $D$ is the union of a set of non-comparable elementary regions, along with all elementary regions less than these under the partial order described above. The* weight $\mathbf{w}(R)$ *of a coherent region $R$ is twice the number of 4-valent vertices plus the number of bivalent vertices in the closure of $R$. The* non-local relation associated to $R$ *is*

$$t^{\mathbf{w}(R)}w_{out} - w_{in},$$

*where $w_{out}$ (respectively $w_{in}$) is the product of the edges outside $R$, but incident to exactly one vertex of $\partial R$ and pointing out from (respectively into) $R$.*

There are five coherent regions in the singularized figure 8 example of Figure 3, with associated relations as follows. Notice that, for example, $E_1 \cup E_2 \cup E_4$ is not a coherent region because $E_3 < E_4$.

| coherent region | non-local relation |
|---|---|
| $E_1$ | $t^6x_1x_7 - x_4x_{10}$ |
| $E_1 \cup E_2$ | $t^8x_1x_9 - x_4x_6$ |
| $E_1 \cup E_3$ | $t^8x_3x_7 - x_0x_{10}$ |
| $E_1 \cup E_2 \cup E_3$ | $t^{10}x_3x_9 - x_0x_6$ |
| $E_1 \cup E_2 \cup E_3 \cup E_4$ | $t^{11}x_9 - x_0$ |

Finally, we may think of non-local relations as arising from subsets of vertices in the $I$-resolution of $D$.

**Definition 3** (Subsets)**.** *Let $V$ be a subset of the vertices in the $I$-resolution of $D$. The* weight $\mathbf{w}(V)$ *of $V$ is twice the number of 4-valent vertices plus the number of bivalent vertices in $V$. The* non-local relation associated to $V$ *is*

$$t^{\mathbf{w}(V)}w_{out} - w_{in},$$

*where $w_{out}$ is the product of edges from $V$ to its complement and $w_{in}$ is the product of edges into $V$ from its complement.*

Any of these three definitions gives a generating set for $\mathcal{N}_I(D)$. We will prove that the three definitions are equivalent in Proposition 1. First, we record some observations about the efficiency of the generating sets prescribed by the different definitions.

A priori, the generating set obtained from subsets is much larger than those obtained from cycles or coherent regions. However, it actually suffices to consider a smaller collection of subsets whose associated relations still generate the same ideal in $\mathcal{R}[x_0,\ldots,x_n]/\mathcal{L}$. First, we may restrict to connected subsets of vertices, meaning those whose union with their incident edges is a connected graph. If a subset $V$ is disconnected as $V = V' \coprod V''$, then the outgoing (respectively incoming) edges from $V$ are exactly the union of the incoming (respectively outgoing) edges from $V'$ and $V''$. Therefore, the non-local relation associated to $V$ has the form

$$t^{\mathbf{w}(V')+\mathbf{w}(V'')} w'_{\text{out}} w''_{\text{out}} - w'_{\text{in}} w''_{\text{in}}.$$

However, this is already contained in the ideal generated by

$$t^{\mathbf{w}(V')} w'_{\text{out}} - w'_{\text{in}} \quad \text{and} \quad t^{\mathbf{w}(V'')} w''_{\text{out}} - w''_{\text{in}},$$

which are the non-local relations associated to $V'$ and $V''$.

Second, we may ignore a subset $V$ if the union of $V$ with its incident edges is a graph with no oriented cycles. In Figure 3, the two vertices in layer $s_0$ along with the 4-valent vertex in layer $s_1$ form such a subset. The non-local relation associated to this subset is $t^5 x_3 x_7 x_8 - x_0 x_1 x_2$, but simple substitutions using the local relations associated to the three vertices in the subset show that this supposedly non-local relation is actually contained in $\mathcal{L}$.

**Observation 1.** *The ideal of non-local relations $\mathcal{N}_I$ can be generated by the non-local relations associated to connected subsets that contain oriented cycles.*

We prove this statement inductively, noting as a base case that the non-local relation associated to a subset with a single vertex is identical to the local relation associated to that vertex. Suppose $V$ is a connected subset with no oriented cycles, that $v$ is a bivalent vertex, and that $V \cup \{v\}$ is a connected subset with no oriented cycles. Then the non-local relation associated to $V \cup \{v\}$ is already contained in the ideal sum of $\mathcal{L}$ with the non-local relation associated to $V$. Suppose that $x_{\text{out}}$ is the edge from $V$ to $v$ and $x_v$ is the edge pointing out from $v$. If $t^{\mathbf{w}}(V) w_{\text{out}} x_{\text{out}} - w_{\text{in}}$ is the non-local relation associated to $V$, then the relation associated to $V \cup \{v\}$ is $t^{\mathbf{w}(V)+1} w_{\text{out}} x_v - w_{\text{in}}$. Using the local relation $t x_v - x_{\text{out}}$ to replace $x_v$ recovers the non-local relation associated to $V$. A similar argument applies if the edge between $V$ and $\{v\}$ is oriented in the opposite direction.

Suppose instead that $v$ is a 4-valent vertex with edges $x_a$ and $x_b$ pointing out and edges $x_c$ and $x_{\text{out}}$ pointing in. Suppose that $x_{\text{out}}$ connects to a vertex $v' \in V$ and that none of $x_a, x_b, x_c$ are incident to any vertex in $V$. The local relation associated to $v$ is then $t^2 x_a x_b - x_c x_{\text{out}}$, while the non-local relation associated to $V$ is of the form $t^{\mathbf{w}(V)} w_{\text{out}} x_{\text{out}} - w_{\text{in}}$. The non-local relation associated to $V \cup \{v\}$ is

$$t^{\mathbf{w}(V)+2} w_{\text{out}} x_a x_b - w_{\text{in}} x_c \equiv t^{\mathbf{w}(V)} w_{\text{out}} x_c x_{\text{out}} - w_{\text{in}} x_c = x_c (t^{\mathbf{w}(V)} w_{\text{out}} x_{\text{out}} - w_{\text{in}}).$$

Therefore, extending a connected graph with no oriented cycles by an adjacent 4-valent vertex produces a non-local relation already contained in the ideal sum of $\mathcal{L}$ with the non-local relation associated to $V$.

The second observation of this section concerns redundancy in the generating sets for $\mathcal{N}_I$ defined by cycles and coherent regions arising from certain elementary regions that can be removed from a coherent region without producing an independent non-local relation. For instance, in Figure 3, the coherent region $E_1 \cup E_2 \cup E_3 \cup E_4$ specifies the non-local relation $t^{11}x_9 - x_0$ as a generator for $\mathcal{N}_I$. Then $x_6(t^{11}x_9 - x_0)$ is also in $\mathcal{N}_I$. It can be modified to $t^{10}x_3x_9 - x_0x_6$ using the relation $tx_6 - x_3$, which is the linear relation associated to the bivalent vertex in layer $s_1$. We have obtained the non-local relation associated to $E_1 \cup E_2 \cup E_3$, showing that it is redundant once the non-local relation for $E_1 \cup E_2 \cup E_3 \cup E_4$ is included in the generating set of $\mathcal{N}_I$. More formally, we have the following observation.

**Observation 2.** *Suppose a coherent region $R$ has an adjacent elementary region $E$ and that $\partial E \setminus \partial R \cap \partial E$ is a path of edges through bivalent vertices only. Then the non-local relation associated to $R$ is contained in the ideal sum of $\mathcal{L}$ with the non-local relation associated to $R \cup E$.*

Label the edges in the path in $\partial E \setminus \partial R \cap \partial E$ by $x_{\text{out}}, x_1, \dots, x_p, x_{\text{in}}$ consistent with the orientation of the overall diagram. The linear relations associated to each vertex in this path are $tx_1 - x_{\text{out}}$, $tx_{\text{in}} - x_p$, and $tx_{i+1} - x_i$ for $1 \le i \le p-1$. The non-local relation associated to $R$ has the form

$$t^{\mathbf{w}}(R)w_{\text{out}}x_{\text{out}} - w_{\text{in}}x_{\text{in}},$$

which can be rewritten using the linear relations above to give

$$t^{\mathbf{w}(R)+p+1}w_{\text{out}}x_{\text{in}} - w_{\text{in}}x_{\text{in}} = \left(t^{\mathbf{w}(R)+p+1}w_{\text{out}} - w_{\text{in}}\right)x_{\text{in}},$$

which is a multiple of the non-local relation associated to $R \cup E$. Therefore, to form a minimal generating set for $\mathcal{N}_I$, we need only consider $R \cup E$.

As these observations begin to indicate, the definitions of non-local relations via cycles, coherent regions, and subsets are equivalent. In the example of Figure 3, the cycle shown in bold produces the same non-local relation as the coherent region $E_1 \cup E_2$ or the subset of vertices contained in $E_1 \cup E_2$. These correspondences between cycles, coherent regions, and subsets hold in general.

**Proposition 1.** *Definitions 1, 2, and 3 produce the same ideal in $\mathcal{R}[x_0, \dots, x_n]/\mathcal{L}$, where $\mathcal{L}$ is the ideal generated by local relations associated to each vertex in the $I$-resolution of $D$.*

*Proof.* The equivalence between definitions 1 (cycles) and 2 (coherent regions) is clear: the boundaries of coherent regions are exactly the cycles that avoid the basepoint and have orientations matching that of $D$. (Consider, for example, the boundary of $E_1 \cup E_2 \cup E_3$ compared to that of $E_1 \cup E_2 \cup E_4$ in Figure 3.) Weights and the edge products $w_{\text{out}}$ and $w_{\text{in}}$ are identical for a coherent region $R$ and the cycle $\partial R$, so the associated non-local relations are the same.

Let $\mathcal{N}$ denote the ideal generated by non-local relations associated to cycles or coherent regions in $\mathcal{R}[\underline{x}(D)]/\mathcal{L}$. Let $\mathcal{N}_S$ denote the ideal generated by non-local relations associated to subsets. Suppose $R$ is a coherent region and $V_R$ the set of vertices in its closure. Then $\mathbf{w}(R) = \mathbf{w}(V_R) = \mathbf{w}(\partial R)$ and the words $w_{\text{out}}$ and $w_{\text{in}}$ defined with respect to $R$, $\partial R$, or $V_R$ are the same. Therefore, we have the inclusion $\mathcal{N} \subset \mathcal{N}_S$.

For the opposite inclusion, consider a subset $V$. We appeal first to Observation 1, which allows us to assume that the union of $V$ and its incident edges forms a

connected graph containing an oriented cycle $Z$. Assume that $Z$ is the outermost cycle contained in $V$, and let $R_Z$ be the coherent region it bounds. If $V$ contains all of the vertices in the closure of $R_Z$, then the arguments about connectedness and subsets just after Observation 1 allow us to remove all vertices from $V$ that are not contained in the closure of $R_Z$, thereby showing that the non-local relation associated to $V$ can be constructed from the non-local relation associated to $R_Z$.

Suppose now that $V$ does not contain all of the vertices in $R_Z$. Then the complement of $V$ is disconnected, with one component inside $Z$ and one component outside. Denote these components $V'$ and $V''$, respectively. Then $V \cup V'$ contains $Z$ and all of the vertices in the closure of $R_Z$, so the argument above shows that its associated non-local relation is contained in $\mathcal{N}$. The subset $V'$ may not contain any oriented cycles or it may contain an oriented cycle $Z'$ and all vertices in the closure of $R_{Z'}$. Therefore, its associated non-local relation is contained in either $\mathcal{L}$ or $\mathcal{N}$.

Finally, we show that the non-local relation associated to $V$ is in the ideal generated by the non-local relations associated to $V'$ and $V \cup V'$. The words $w_{\text{out}}$ and $w_{\text{in}}$ defined with respect to $V$ are products $w_{\text{out}} = w'_{\text{in}} w''_{\text{in}}$ and $w_{\text{in}} = w'_{\text{out}} w''_{\text{out}}$ of edges into and out from $V'$ and $V''$.

$$\begin{aligned} & t^{\mathbf{w}(V)} w'_{\text{in}} w''_{\text{in}} - w'_{\text{out}} w''_{\text{out}} && \text{non-local relation from } V \\ \equiv\ & t^{\mathbf{w}(V)+\mathbf{w}(V')} w'_{\text{out}} w''_{\text{in}} - w'_{\text{out}} w''_{\text{out}} && \text{by substituting non-local relation from } V' \\ =\ & (t^{\mathbf{w}(V)+\mathbf{w}(V')} w''_{\text{in}} - w''_{\text{out}}) w'_{\text{out}} && \text{a multiple of the non-local relation } V \cup V' \end{aligned}$$

Since the non-local relation associated to $V$ can be constructed from those associated to $V'$ and $V \cup V'$, it is contained in $\mathcal{N}$. Therefore, any non-local relation associated to a subset can be generated from non-local relations associated to coherent regions, meaning that $\mathcal{N}_S \subset \mathcal{N}$. □

Two further observations related to the non-local relations are worth recording for later use.

**Observation 3.** *The relation $t^{\mathbf{w}(D)} x_n - x_0$, where $x_n$ is the edge entering the basepoint and $x_0$ is the edge leaving it, holds in $\mathcal{A}_I(D)$ for any $I$ and any $D$. It is associated to the subset containing all vertices or the outermost cycle in the diagram that does not pass through the basepoint.*

**Observation 4.** *If $I$ is a disconnected resolution of $D$, and we choose to work over a completed ground ring, then the algebra associated to the $I$-resolution of $D$ will vanish. In a disconnected resolution, there are cycles that do not contain the basepoint and have no ingoing or outgoing edges. In this situation, we interpret the products $w_{out}$ and $w_{in}$ to be $1$, which makes the associated relation $t^k - 1$ for some $k$. In $\widehat{\mathcal{R}}$ or $\widehat{\mathcal{R}[\underline{x}]}$, $t^k - 1$ is a unit. Therefore, including $t^k - 1$ in our ideal of relations makes $\mathcal{A}_I(D) \otimes_{\mathcal{R}} \widehat{\mathcal{R}}$ or $\mathcal{A}_I(D) \otimes_{\mathcal{R}[\underline{x}]} \widehat{\mathcal{R}[\underline{x}]}$ vanish.*

## 4. Removing bivalent vertices

This section is devoted to a technical result allowing us to remove a horizontal layer of a diagram with a bivalent vertex on each strand and no 4-valent vertices. Such a layer is obtained each time a crossing is resolved. Suppose the $I$-resolution of $D$ is a diagram with $\ell+1$ layers, and that layer $k$ contains only bivalent vertices. Let $\overline{D}$ denote the diagram obtained by removing layer $k$. The proposition below shows

that removing layer $k$ corresponds to tensoring $\mathcal{A}_I(D)$ with the ground ring via a non-trivial automorphism. Note that applying this base change to every summand of the chain complex $C(D)$ does not change the homology of the complex, since $\mathcal{R}$ is flat when considered as an $\mathcal{R}$-module via an automorphism. We refer to the notation in Figure 4 throughout.

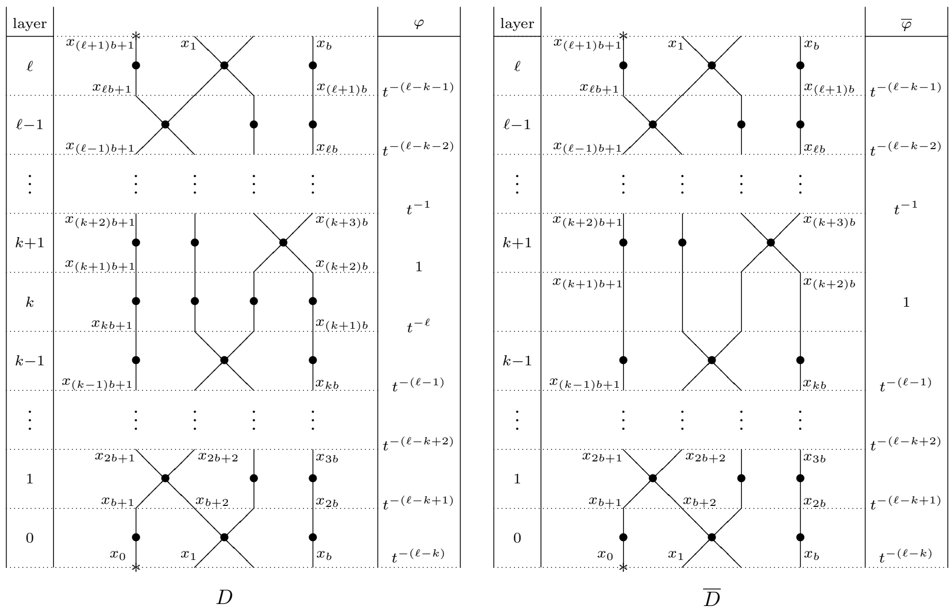


FIGURE 4. Diagrams for the proof of Proposition 2. The maps $\varphi$ on $\mathcal{A}_I(D)$ and $\overline{\varphi}$ on $\mathcal{A}_{\overline{I}}(\overline{D})$ are defined to be multiplication by the factor shown in the right-most column of each diagram.

**Proposition 2.** *Let $D$ and $\overline{D}$ be defined as above. Let $\overline{I}$ denote the index $I$ with its $k^{th}$ component deleted. Then there is an $\mathcal{R}[\underline{x}(\overline{D})]$-module isomorphism $\mathcal{A}_{\overline{I}}(\overline{D}) \cong \mathcal{A}_I(D) \otimes_{(\mathcal{R},\psi)} \mathcal{R}$, where $\psi$ is the automorphism of $\mathcal{R}$ taking $1$ to $1$ and $t$ to $t^{\ell/(\ell+1)}$.*

*Proof.* We first define automorphisms $\varphi$ of $\mathcal{A}_I(D)$ and $\overline{\varphi}$ of $\mathcal{A}_I(\overline{D})$ that transform our original presentations of these algebras into presentations in which $t$ appears very rarely. That $\psi$ is the necessary automorphism of $\mathcal{R}$ will then be apparent.

Define $\varphi$ to be multiplication by $t^{-(j-1)}$ on edges $x_{(k+j)b+i}$ for $0 \leq j \leq \ell$ and $1 \leq i \leq b$ (treating the $k+j$ portion of the subscript modulo $\ell+1$), and multiplication by $t^{-(\ell-k)}$ on edge $x_{(\ell+1)b+1} = x_n$. That is, $\varphi$ is the identity on the edges connecting layer $k$ to layer $k+1$ (edges $x_{(k+1)b+1}, \ldots, x_{(k+2)b}$), multiplication by $t^{-1}$ on the edges connecting layer $k+1$ to layer $k+2$ (edges $x_{(k+2)b+1}, \ldots, x_{(k+3)b}$), multiplication by $t^{-2}$ on the edges connecting layer $k+2$ to layer $k+3$, and so on, until it is multiplication by $t^{-\ell}$ on the edges connecting layer $k-1$ to layer $k$ (edges $x_{kb+1}, \ldots, x_{(k+1)b}$).

We may continue to use $x_0, \ldots, x_n$ as generators of $\varphi(\mathcal{A}_I(D))$, but must examine carefully the effect of $\varphi$ on the generating sets of $\mathcal{L}$ and $\mathcal{N}_I(D)$. Consider first the

generators of $\mathcal{L}_{k+j}$ for any $j \neq 0$. These have one of the following forms, where $1 \leq i \leq b$.

$$tx_{(k+j+1)b+i} + tx_{(k+j+1)b+i+1} - x_{(k+j)b+i} - x_{(k+j)b+i+1} \tag{3}$$

$$t^2 x_{(k+j+1)b+i} x_{(k+j+1)b+i+1} - x_{(k+j)b+i} x_{(k+j)b+i+1} \tag{4}$$

$$tx_{(k+j+1)b+i} - x_{(k+j)b+i} \tag{5}$$

After applying $\varphi$, they become

$$\begin{aligned} t^{-j+1}&x_{(k+j+1)b+i} + t^{-j+1}x_{(k+j+1)b+i+1} - t^{-(j-1)}x_{(k+j)b+i} - t^{-(j-1)}x_{(k+j)b+i+1} \\ &\equiv x_{(k+j+1)b+i} + x_{(k+j+1)b+i+1} - x_{(k+j)b+i} - x_{(k+j)b+i+1} \end{aligned} \tag{6}$$

$$\begin{aligned} t^{-2j+2}&x_{(k+j+1)b+i}x_{(k+j+1)b+i+1} - t^{-2(j-1)}x_{(k+j)b+i}x_{(k+j)b+i+1} \\ &\equiv x_{(k+j+1)b+i}x_{(k+j+1)b+i+1} - x_{(k+j)b+i}x_{(k+j)b+i+1} \end{aligned} \tag{7}$$

$$\begin{aligned} t^{-j+1}&x_{(k+j+1)b+i} - t^{-(j-1)}x_{(k+j)b+i} \\ &\equiv x_{(k+j+1)b+i} - x_{(k+j)b+i}. \end{aligned} \tag{8}$$

The price of eliminating powers of $t$ from most local relations is that $t$ appears with higher powers in relations that do involve layer $k$. Since layer $k$ has only bivalent vertices, its associated relations are all of the form $tx_{(k+1)b+i} - x_{kb+i}$. Applying $\varphi$, we obtain

$$tx_{(k+1)b+i} - t^{-\ell}x_{kb+i} \equiv t^{\ell+1}x_{(k+1)b+i} - x_{kb+i}.$$

Non-local relations are similarly affected. Consider the generating set for $\mathcal{N}_I(D)$ given by coherent regions. We will show that $\varphi$ applied to any relation in this generating set produces a relation of the form $t^{p(\ell+1)}w_{\text{out}} - w_{\text{in}}$ for some integer $p$. Begin with the innermost elementary region $E_1$. Suppose it has $v$ 4-valent vertices along its boundary in layers $k+j_1, \ldots, k+j_v$. Then $\mathbf{w}(E_1) = \ell + 1 + v$. Each 4-valent vertex contributes one edge to the product $w_{\text{out}}$ and an edge one layer lower to $w_{\text{in}}$. If $j_i \neq 0$ for $1 \leq i \leq v$, then

$$\begin{aligned} &\varphi\left(w_{\text{out}}\right) = t^{-j_1-\cdots-j_v}w_{\text{out}} && \text{and} \\ &\varphi\left(w_{\text{in}}\right) = t^{-(j_1-1)-\cdots-(j_v-1)}w_{\text{in}} = t^{-j_1-\cdots-j_v+v}w_{\text{in}}, && \text{so} \\ &\varphi\left(t^{\ell+1+v}w_{\text{out}} - w_{\text{in}}\right) \equiv t^{\ell+1}w_{\text{out}} - w_{\text{in}}. \end{aligned}$$

Suppose instead (without loss of generality) that $j_1 = 0$. Then $\varphi$ is the identity when applied to the outgoing edge of the vertex in layer $k + j_1$, but multiplication by $t^{-\ell}$ on the incoming edge. Therefore,

$$\begin{aligned} &\varphi\left(w_{\text{out}}\right) = t^{-j_2-\cdots-j_v}w_{\text{out}} && \text{and} \\ &\varphi\left(w_{\text{in}}\right) = t^{-\ell-(j_2-1)-\cdots-(j_v-1)}w_{\text{in}} = t^{-\ell-j_2-\cdots-j_v+v-1}w_{\text{in}}, && \text{so} \\ &\varphi\left(t^{\ell+1+v}w_{\text{out}} - w_{\text{in}}\right) \equiv t^{2(\ell+1)}w_{\text{out}} - w_{\text{in}}. \end{aligned}$$

So $\varphi$ has the claimed effect on the non-local relation associated to the innermost coherent region.

Next consider an elementary region $E \neq E_1$ with bottom-most vertex in layer $k+j$ and top-most vertex in layer $k+j+s$. Suppose $\partial E$ meets $v'$ additional 4-valent vertices in layers $k+j_1, \ldots, k+j_{v'}$. Assume for now that $E$ does not meet layer $k$. Then $\mathbf{w}(E) = 2(s+1) + v'$. Let $t^{2(s+1)+v'}e_{\text{out}} - e_{\text{in}}$ denote the non-local relation associated to $E$. The top-most vertex of $E$ contributes two outgoing edges to $e_{\text{out}}$

and the bottom-most vertex contributes two incoming edges to $e_{\text{in}}$. The other $v'$ 4-valent vertices contribute one edge each to $e_{\text{out}}$ and $e_{\text{in}}$. Therefore,

$$\begin{aligned}
&\varphi\left(e_{\text{out}}\right)=t^{-2(j+s)-j_1-\cdots-j_{v'}}e_{\text{out}}=t^{-2j-j_1-\cdots-j_{v'}-2s}e_{\text{out}} && \text{and}\\
&\varphi\left(e_{\text{in}}\right)=t^{-2(j-1)-(j_1-1)-\cdots-(j_{v'}-1)}e_{\text{in}}=t^{-2j-j_1-\cdots-j_{v'}+v'+2}e_{\text{in}}, && \text{so}\\
&\varphi\left(t^{2(s+1)+v'}e_{\text{out}}-e_{\text{in}}\right)\equiv e_{\text{out}}-e_{\text{in}}.
\end{aligned}$$

If $E$ does meet layer $k$, a then modification of the calculation above (similar to that used for $E_1$) verifies the claim that $\varphi\left(t^{2(s+1)+v'}e_{\text{out}}-e_{\text{in}}\right)$ has the form $t^{p(\ell+1)}e_{\text{out}}-e_{\text{in}}$ for some integer $p$.

Finally, consider a coherent region $R'$ that is not elementary. We can write $R'$ as $R\cup E$, where $R$ is a coherent region and $E$ is an elementary region. Suppose the non-local relations associated to $R$ and $E$ are $t^{\mathbf{w}(R)}w_{\text{out}}-w_{\text{in}}$ and $t^{\mathbf{w}(E)}e_{\text{out}}-e_{\text{in}}$, respectively. Let $y$ be the product of edges that connect vertices in $R$ to vertices in $E$. The non-local relation associated to $R'$ can be obtained by combining the non-local relations associated to $R$ and $E$, then factoring out $y$ as follows.

$$t^{\mathbf{w}(R)+\mathbf{w}(E)}w_{\text{out}}e_{\text{out}}-w_{\text{in}}e_{\text{in}}=y\left(t^{\mathbf{w}(R)+\mathbf{w}(E)}w'_{\text{out}}e'_{\text{out}}-w'_{\text{in}}e'_{\text{in}}\right)$$

The non-local relation associated to $R'$ is $t^{\mathbf{w}(R)+\mathbf{w}(E)}w'_{\text{out}}e'_{\text{out}}-w'_{\text{in}}e'_{\text{in}}$. We will assume inductively that $\varphi$ applied to the non-local relations for $R$ and $E$ produces $t^{p(\ell+1)}w_{\text{out}}-w_{\text{in}}$ and $t^{q(\ell+1)}e_{\text{out}}-e_{\text{in}}$, respectively for some integers $p$ and $q$. Then

$$\begin{aligned}
&\varphi\left(t^{\mathbf{w}(R)+\mathbf{w}(E)}w_{\text{out}}e_{\text{out}}-w_{\text{in}}e_{\text{in}}\right)\\
&\equiv t^{(p+q)(\ell+1)}w_{\text{out}}e_{\text{out}}-w_{\text{in}}e_{\text{in}}\\
&=y\left(t^{(p+q)(\ell+1)}w'_{\text{out}}e'_{\text{out}}-w'_{\text{in}}e'_{\text{in}}\right)
\end{aligned}$$

and on the other hand

$$\begin{aligned}
&\varphi\left(t^{\mathbf{w}(R)+\mathbf{w}(E)}w_{\text{out}}e_{\text{out}}-w_{\text{in}}e_{\text{in}}\right)\\
&\equiv\varphi(y)\,\varphi\left(t^{\mathbf{w}(R)+\mathbf{w}(E)}w'_{\text{out}}e'_{\text{out}}-w'_{\text{in}}e'_{\text{in}}\right)\\
&\equiv y\,\varphi\left(t^{\mathbf{w}(R)+\mathbf{w}(E)}w'_{\text{out}}e'_{\text{out}}-w'_{\text{in}}e'_{\text{in}}\right).
\end{aligned}$$

We have verified that applying $\varphi$ to the non-local relation associated to $R'$ produces a relation in which the power of $t$ is an integer multiple of $\ell+1$.

So far, we have relations of the following forms in our presentation of $\varphi(\mathcal{A}_I(D))$.

$$\begin{gathered}
x_{(k+j+1)b+i}+x_{(k+j+1)b+i+1}-x_{(k+j)b+i}-x_{(k+j)b+i+1}\\
x_{(k+j+1)b+i}x_{(k+j+1)b+i+1}-x_{(k+j)b+i}x_{(k+j)b+i+1}\\
x_{(k+j+1)b+i}-x_{(k+j)b+i}\\
t^{\ell+1}x_{(k+1)b+i}-x_{kb+1}\\
t^{p(\ell+1)}w_{\text{out}}-w_{\text{in}}
\end{gathered}\tag{9}$$

It will be convenient to make one final modification: use the relations in (9) to eliminate the variables for edges connecting layer $k-1$ to layer $k$. The result is a presentation in which $t$ appears only in the following types of relations.

$$t^{\ell+1}x_{(k+1)b+i}+t^{\ell+1}x_{(k+1)b+i+1}-x_{(k-1)b+i}-x_{(k-1)b+i+1} \tag{10}$$

$$t^{2(\ell+1)}x_{(k+1)b+i}x_{(k+1)b+i+1}-x_{(k-1)b+i}x_{(k-1)b+i+1} \tag{11}$$

$$t^{\ell+1}x_{(k+1)b+i}-x_{(k-1)b+i} \tag{12}$$

$$t^{p(\ell+1)}w_{\text{out}}-w_{\text{in}} \tag{13}$$

The second map, $\overline{\varphi}$, allows us to present $\mathcal{A}_{\overline{I}}(\overline{D})$ in a similar way, with powers of $t$ appearing only in certain relations, and only as $t^{p\ell}$ for various integers $p$. Define $\overline{\varphi}$ in exactly the same way as $\varphi$ on edges $x_{(k+j)b+i}$ for $1 \leq j \leq \ell$ and $0 \leq i \leq b$ and for edge $x_{(\ell+1)b+1}$. Diagram $\overline{D}$ has no $k^{th}$ layer, so $\overline{\varphi}$ is the identity on the edges connecting layer $k-1$ to layer $k+1$, multiplication by $t^{-1}$ on the edges connecting layer $k+1$ to layer $k+2$, multiplication by $t^{-2}$ on the edges connecting layer $k+2$ to $k+3$, and so on, until it is multiplication by $t^{-(\ell-1)}$ on the edges connecting layer $k-2$ to layer $k-1$.

Again, for most relations, $\overline{\varphi}$ eliminates all powers of $t$. Similar calculations to those above show that $\overline{\varphi}$ removes $t$ from the generating set for $\mathcal{L}_{k+j}$ for $j \neq 0$, leaving relations identical to those in (6) to (8) above.

All powers of $t$ end up in generators of $\mathcal{L}_{k-1}$ and $\mathcal{N}_I$, but this time with multiples of $\ell$ instead of $\ell+1$. The relations that involve $t$ have one of the following forms.

$$t^{\ell}x_{(k+1)b+i}+t^{\ell}x_{(k+1)b+i+1}-x_{(k-1)b+i}-x_{(k-1)b+i+1} \tag{14}$$

$$t^{2\ell}x_{(k+1)b+i}x_{(k+1)b+i+1}-x_{(k-1)b+i}x_{(k-1)b+i+1} \tag{15}$$

$$t^{\ell}x_{(k+1)b+i}-x_{(k-1)b+i} \tag{16}$$

$$t^{p\ell}w_{\text{out}}-w_{\text{in}} \tag{17}$$

We now have presentations of $\mathcal{A}_I(D)$ and $\mathcal{A}_{\overline{I}}(\overline{D})$, both over the smaller edge ring $\mathcal{R}[\underline{x}(\overline{D})]$, that differ only by whether $t$ appears with a power of $\ell+1$ or with $\ell$. The map needed to relate these two presentations is an automorphism of $\mathcal{R}$. Define $\psi : \mathcal{R} \to \mathcal{R}$ to take 1 to 1 and $t$ to $t^{\ell/(\ell+1)}$. Applying $\psi$ to the relations in (10)-(13) produces exactly the relations in (14)-(17). Since no other relations in our presentation of $\mathcal{A}_I(D)$ involve $t$, $\psi$ has no effect on them. Therefore, $\varphi(\mathcal{A}_I(D)) \otimes_{(\mathcal{R},\psi)} \mathcal{R}$ and $\overline{\varphi}(\mathcal{A}_{\overline{I}}(\overline{D}))$ have identical presentations as $\mathcal{R}[\underline{x}(\overline{D})]$-modules. $\square$

## 5. Braid-like Reidemeister Move II

Suppose $D$ and $\overline{D}$ are two knot projections that differ by a Reidemeister II move with labels as in Figure 5. The edge rings of $D$ and $\overline{D}$ are related by $\mathcal{R}[\underline{x}(D)] = \mathcal{R}[\underline{x}(\overline{D})][x_3, x_4, x_5, x_6]$. We will show that $C(D)$ and $C(\overline{D})$ are chain homotopy equivalent as complexes of $\mathcal{R}[\underline{x}(\overline{D})]$-algebras, but will work over the larger edge ring $\mathcal{R}[\underline{x}(D)]$ for as long as possible. Throughout this section, we will abbreviate indices of resolutions to two entries, showing only the states of the crossings in layers $s_i$ and $s_{i+1}$.

There are two oriented Reidemeister II moves, depending on which crossing in $D$ is positive and which is negative, but the arguments are very similar in the two cases. The relevant portion of $C(D)$ is shown in Figure 6. The two variants of the Reidemeister II move exchange $\mathcal{A}_{00}(D)$ with $\mathcal{A}_{11}(D)$ and $\mathcal{A}_{01}(D)$ with $\mathcal{A}_{10}(D)$.

The key step in proving that the chain homotopy type of $C(D)$ is unchanged by a Reidemeister II move is to show the equivalence of the two complexes in Figure 6. It suffices to prove the statement in Lemma 3, which asserts that (after a change of basis) $\mathcal{A}_{00}(D) \xrightarrow{f} \mathcal{A}_{10}(D) \xrightarrow{g} \mathcal{A}_{11}(D)$ is an acyclic subcomplex. Removing that subcomplex leaves the bottom complex of Figure 6. Removing the bivalent vertices in layers $s_i$ and $s_{i+1}$ (applying Proposition 2 and reverting to the edge ring $\mathcal{R}[\underline{x}(\overline{D})]$) leaves the corresponding portion of $C(\overline{D})$.

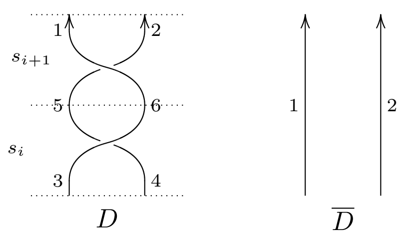



FIGURE 5. Projection $D$ layers $s_i$ and $s_{i+1}$ and the corresponding portion of $\overline{D}$, which has no vertices. Technically, $\overline{D}$ does not have layers corresponding to $s_i$ and $s_{i+1}$; it is identical to $D$ in all other layers. Assume that the braid axis is to the right of each diagram.

**Lemma 3.** *As $\mathcal{R}[\underline{x}(\overline{D})][x_3, x_4]$-modules, $\mathcal{A}_{10}(D) \cong \mathcal{A}_{00}(D) \oplus \mathcal{A}_{11}(D)$, $f$ is an isomorphism onto the first summand, and $g$ is an isomorphism when restricted to the second summand.*

*Proof.* The following matrix is a generating set for $\mathcal{L}_i + \mathcal{L}_{i+1}$ in the 10-resolution of $D$.

$$\begin{pmatrix} t(x_1+x_2)-(x_5+x_6) \\ (tx_1-x_6)(tx_2-x_6) \\ t(x_5+x_6)-(x_3+x_4) \\ (tx_6-x_4)(x_3-tx_6) \end{pmatrix}$$

Use row I to eliminate $x_5$, then rewrite to limit the appearance of $x_6$ to a single row.

$$\begin{pmatrix} (tx_1-x_6)(tx_2-x_6) \\ t^2(x_1+x_2)-(x_3+x_4) \\ (tx_6-x_4)(x_3-tx_6) \end{pmatrix} \xrightarrow{\mathrm{III}+t^2\mathrm{I}+tx_6\mathrm{II}} \begin{pmatrix} (tx_1-x_6)(tx_2-x_6) \\ t^2(x_1+x_2)-(x_3+x_4) \\ t^4x_1x_2-x_3x_4 \end{pmatrix}$$

Let $\overline{\mathcal{L}}$ denote the ideal generated by the last two rows of the matrix above and $\mathcal{L}$ denote the ideal generated by local relations in layers other than $i$ and $i+1$. Note that $x_5$ and $x_6$ do not appear in the generating set for $\mathcal{L}$. By Observation 2, they need not appear in a generating set for $\mathcal{N}_{10}$ either. Therefore, these ideals survive the manipulations above unchanged. Define

$$\mathcal{S} = \frac{\mathcal{R}[x_0,\ldots,x_4,x_7,\ldots,x_n]}{\overline{\mathcal{L}}+\mathcal{L}+\mathcal{N}_{10}}.$$

We have simplified the presentation of $\mathcal{A}_{10}$ so that $x_6$ appears only in one relation, which is quadratic in $x_6$. Using that relation, we may split $\mathcal{A}_{10}$ as follows.

$$\mathcal{A}_{10}(D) \cong \frac{\mathcal{S}[x_6]}{(tx_1-x_6)(tx_2-x_6)} \cong \mathcal{S}(1) \oplus \mathcal{S}(tx_1-x_6)$$

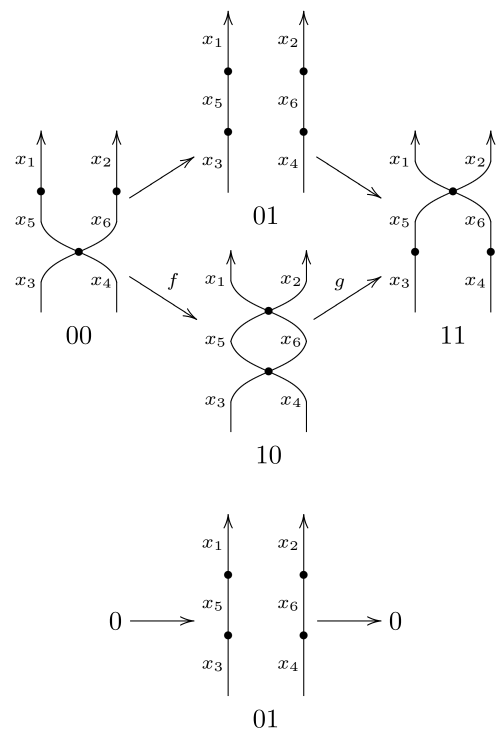



FIGURE 6. The top chain complex is a portion of $C(D)$. Lemma 3 shows that it is chain homotopy equivalent to the bottom chain complex. Assume that the braid axis is to the right of each diagram.

It remains to show that these two summands correspond to $\mathcal{A}_{11}(D)$ and $\mathcal{A}_{00}(D)$.

In the 11-resolution, the linear relations $tx_5 - x_3$ and $tx_6 - x_4$ may be used to replace $x_5$ and $x_6$ throughout the presentation. The resulting local relations in layers $i$ and $i+1$ exactly match those in $\overline{\mathcal{L}}$. The definition by coherent regions and Observation 2 give matching generating sets for $\mathcal{N}_{10}$ and $\mathcal{N}_{11}$. Therefore, $\mathcal{A}_{11}(D)$ has a presentation identical to that of $\mathcal{S}$ given above. Since $g$ is defined to be the quotient map, it is an isomorphism when restricted to the first summand of $\mathcal{A}_{10}(D)$ above.

Similarly, in the 00-resolution, the linear relations $tx_1 - x_5$ and $tx_2 - x_6$ can be used to replace $x_5$ and $x_6$ throughout the presentation of $\mathcal{A}_{00}(D)$. For local relations in layers $i$ and $i+1$, the resulting ideal is exactly $\overline{\mathcal{L}}$. For non-local relations, the definition by coherent regions along with Observation 2 again gives the same generating set for $\mathcal{N}_{00}$ as for $\mathcal{N}_{10}$. Therefore, $\mathcal{A}_{00}(D)$ has a presentation identical to $\mathcal{S}$. Since $f$ is defined to be multiplication by $tx_1 - x_6$, it is an isomorphism onto the second summand of $\mathcal{A}_{10}(D)$ above. $\square$

## 6. Conjugation: Moving the basepoint

In this section we demonstrate that $\mathcal{A}_I(D)$ is invariant under conjugation of the braid diagram $D$. Since conjugation is a planar isotopy of a braid diagram, it does not change the edge ring or the local relations. However, our construction in Section 2 does rely on the choice of a basepoint, the special marking $*$, which has a role in determining which cycles, subsets, or regions are used to define non-local relations. Proving that the algebra $\mathcal{A}_I(D)$ is invariant under conjugation is equivalent to proving that it is invariant under moving the basepoint from one edge to another. Of course, it suffices to simply move the basepoint to an adjacent outermost edge, either past a bivalent vertex or past a singular crossing. Figures 7 and 8 show the two moves we must check.

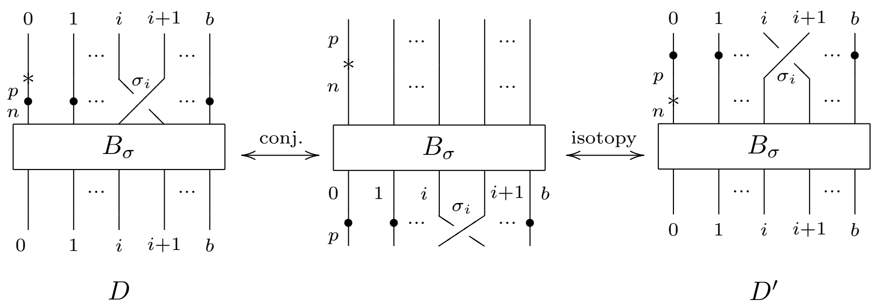



Figure 7. Diagram for Lemma 4: moving the basepoint across a bivalent vertex is equivalent to conjugating $\sigma_i\sigma$ to $\sigma\sigma_i$ for $i \neq 1$.

**Lemma 4.** *Let $D$ be the layered braid diagram for a braid word of the form $\sigma_i\sigma$, where $i \neq 1$ and $\sigma$ is any braid word. Let $D'$ be the layered braid diagram for $\sigma\sigma_i$. Fix edge labels as in Figure 7 with $p > n$. Then for any index $I$, $\mathcal{A}_I(D) \cong \mathcal{A}_I(D')$ as $\mathcal{R}[x]$-algebras, where $x$ acts as the variable associated to the vertex outgoing from the basepoint in each diagram.*

*Proof.* Whether $\sigma_i$ is resolved or singularized in the $I$-resolution of $D$, the isotopy shown on the right indicates that it suffices to prove that we can move the basepoint across a bivalent vertex on the left-most strand. Let $x_0, \dots, x_n, x_p$ denote the variables in the edge ring for $D$ and $y_0, \dots, y_n, y_p$ denote the variables in the edge ring for $D'$. We will view $\mathcal{A}_I(D)$ and $\mathcal{A}_I(D')$ as $\mathcal{R}[x]$-modules by equating $x$ with $x_0$ and with $y_p$, respectively. Define an $\mathcal{R}[x]$-module map $\varphi : \mathcal{A}_I(D) \to \mathcal{A}_I(D')$ by $x_0 \mapsto y_p$, $x_i \mapsto ty_i$ for $1 \leq i \leq n$, and $x_p \mapsto y_n$. To see that it is well-defined and an isomorphism, first notice that $\varphi$ maps the linear relation $tx_p - x_n$ (coming from the bivalent vertex nearest the basepoint in $D$) to 0 in $\mathcal{A}_I(D')$. Now use the relation $tx_p - x_n$ to find a presentation of $\mathcal{A}_I(D)$ in which $x_p$ does not appear. Suppose that $f(x_0, \dots, x_n)$ is one of the relations in this presentation. Then $\varphi(f(x_0, \dots, x_n)) = f(y_p, ty_1, \dots, ty_n) \equiv f(ty_0, ty_1, \dots, ty_n) \equiv f(y_0, y_1, \dots, y_n)$, where "$\equiv$" here means "generates the same ideal in $\mathcal{R}[y_0, \dots, y_n, y_p]/(ty_0 - y_p)$." Since $ty_0 - y_p$ is a relation in $\mathcal{A}_I(D')$ (associated to the bivalent vertex nearest the basepoint), this calculation says that $\varphi$ identifies each relation in the chosen presentation of $\mathcal{A}_I(D)$ with a

relation in $\mathcal{A}_I(D')$. The map defined by $y_i \mapsto t^{-1}x_i$ for $(0 \leq i \leq n)$ and $y_p \mapsto x_0$ is an inverse for $\varphi$, which one can check is well-defined by a similar argument. $\square$

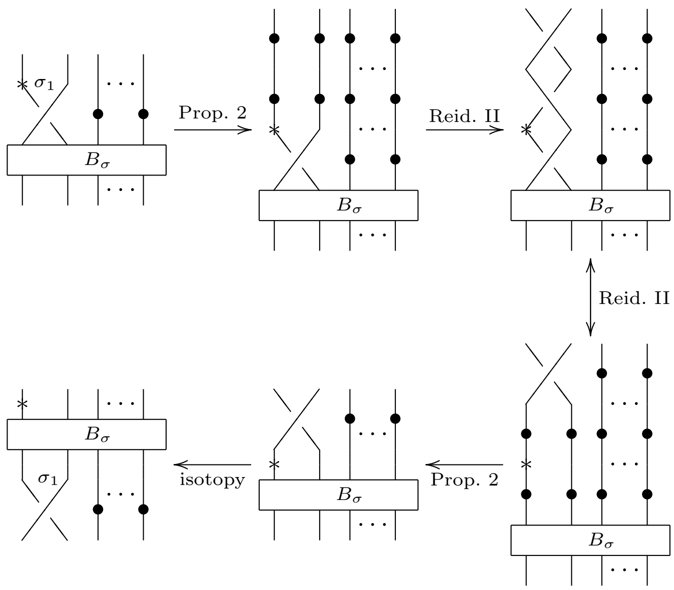



FIGURE 8. Diagrams for Lemma 5: conjugating $\sigma_1\sigma$ to $\sigma\sigma_1$ is equivalent to moving the basepoint across $\sigma_1$.

**Lemma 5.** *Let $D$ be the layered braid diagram for a braid word of the form $\sigma_1\sigma$, where $\sigma$ is any braid word. Let $D'$ be the layered braid diagram for $\sigma\sigma_1$. Fix edge labels as in Figure 8 and let $\underline{x}(B_\sigma)$ denote the edges in $B_\sigma$, including those adjacent to the box labeled $B_\sigma$. Then $C(D)$ and $C(D')$ are equivalent complexes of $\mathcal{R}[\underline{x}(B_\sigma)]$-modules up to chain homotopy equivalence and base change.*

*Proof.* Figure 8 shows a sequence of moves that transforms $D$ on the upper left to $D'$ on the lower left via diagrams $D_1, D_2, D_3, D_4$ moving clockwise around the figure. Each move changes the corresponding chain complex by a base change (Proposition 2) or a chain homotopy equivalence (Lemma 3).

Let $\ell$ be the number of layers in $D$, which is also the number of layers in $D'$. By Proposition 2, $C(D_1) \cong C(D) \otimes_{(\mathcal{R},\psi)} \mathcal{R}$, where $\psi$ is the automorphism of $\mathcal{R}$ that takes $t$ to $t^{(\ell+2)/\ell}$. The next two diagrams are obtained by Reidemeister II moves. Lemma 3 shows that $C(D_2)$ is chain homotopy equivalent to $C(D_1)$.

The Reidemeister II move from $D_2$ to $D_3$ occurs across the basepoint, but Lemma 3 can be modified to apply in this situation. The key step in the modification is to use the relation $t^{\mathbf{w}(D)}x_n - x_0$, where $x_n$ is the edge leaving the basepoint, and $x_0$ is the edge entering the basepoint. As noted in Observation 3, this relation is associated to the outermost cycle, and it holds in any resolution. After using this relation to eliminate $x_n$ from all presentations, the proof of Lemma 3 goes through with only small modifications.

To go from $D_3$ to $D_4$, we remove two layers of bivalent vertices. Proposition 2 implies that $C(D_4) \cong C(D_3) \otimes_{(\mathcal{R},\psi^{-1})} \mathcal{R}$. Finally, $D'$ is obtained by an isotopy, which does not change the chain complex. All together, we have that $C(D')$ is homotopy equivalent to $C(D) \otimes_{(\mathcal{R},\psi)} \mathcal{R} \otimes_{(\mathcal{R},\psi^{-1})} \mathcal{R} \cong C(D)$. $\square$

Although Lemma 5 is stated with an $\mathcal{R}[\underline{x}(B_\sigma)]$-module isomorphism, its proof can be modified to give an $\mathcal{R}[x_0]$-module isomorphism instead (as needed for the proof of Theorem 1) by preserving the edges nearest the basepoint when removing layers of bivalent vertices.

## 7. Braid-like Reidemeister Move III

In this section, we will consider projections $D_1$ and $D_2$ that differ by a Reidemeister III move with all positive crossings and labeling as in Figure 9. The proofs of invariance under the other braid-like versions of Reidmeister III follow similarly because the direct sum splittings we establish below will all be compatible with the appropriate edge maps, whether from a positive or a negative crossing.

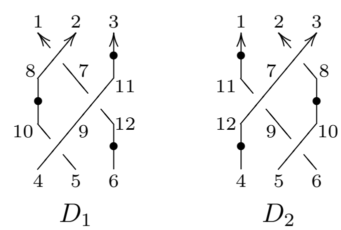



Figure 9. Diagrams $D_1$ and $D_2$ in layers $s_1$, $s_2$, and $s_3$.

Figure 10 shows the relevant portion of the cube of resolutions associated to $D_1$. Throughout this section, we will abbreviate indices to three places, conjugating the diagrams as necessary so that the Reidemeister III move occurs in layers $s_1$, $s_2$, and $s_3$, and using the index to indicate the states of the crossings in those layers only.

The goal is to prove that the chain complexes $C(D_1)$ and $C(D_2)$ are chain homotopy equivalent. The strategy will be to prove that they are each chain homotopy equivalent to the complex following complex, which will be denoted $C(\Upsilon)$.

$$\begin{tikzcd}
 & \mathcal{A}_{100}(D_i) \arrow[r] \arrow[ddr] & \mathcal{A}_{101}(D_i) \arrow[dr] & \\
\mathcal{B} \arrow[ur] \arrow[dr] & & & \mathcal{A}_{111}(D_i) \\
 & \mathcal{A}_{001}(D_i) \arrow[r] \arrow[uur] & \mathcal{A}_{011}(D_i) \arrow[ur] &
\end{tikzcd}$$

The module $\mathcal{B}$ is a direct summand common to $\mathcal{A}_{000}(D_1)$ and $\mathcal{A}_{000}(D_2)$. The other modules in this simplified complex correspond to resolutions of $D_1$ and $D_2$ that are identical after removing extra layers of bivalent vertices. Specifically, notice that the 100-resolution of $D_1$ is isotopic to the 001-resolution of $D_2$ and vice versa; the 101-resolution of $D_1$ is isotopic to the 011-resolution of $D_2$ and vice versa; and the

111-resolutions of $D_1$ and $D_2$ are identical. After removing acyclic subcomplexes, we will find that the complex $C(\Upsilon)$ is common to both $C(D_1)$ and $C(D_2)$.

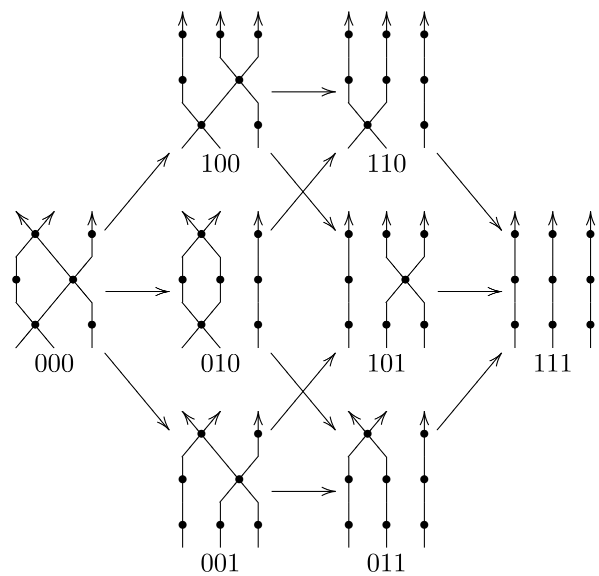



FIGURE 10. Portion of the cube of resolutions for $D_1$ in layers $s_1$, $s_2$, and $s_3$.

Only the argument that $C(D_1)$ is chain homotopy equivalent to $C(\Upsilon)$ will be given here in full detail since the computations needed to establish the same fact about $D_2$ are very similar. The main results of this section are Lemmas 6 and 7, which show that $C(D_1)$ and $C(D_2)$ both have the following form.

$$\begin{tikzcd}
 & \mathcal{A}_{100} \arrow[r] \arrow[dr] & \mathcal{A}_{110} \arrow[dr] & \\
\mathcal{B} \oplus \mathcal{B}_{011} \arrow[ur] \arrow[r] \arrow[dr] & \mathcal{C}_{110} \oplus \mathcal{C}_{011} \arrow[ur] \arrow[dr] & \mathcal{A}_{101} \arrow[r] & \mathcal{A}_{111} \\
 & \mathcal{A}_{001} \arrow[r] \arrow[ur] & \mathcal{A}_{011} \arrow[ur] &
\end{tikzcd}$$

Moreover, these lemmas show that (after a suitable change of basis) there are acyclic subcomplexes $\mathcal{B}_{011} \xrightarrow{1} \mathcal{C}_{011}$ and $\mathcal{C}_{110} \xrightarrow{1} \mathcal{A}_{110}$. After these are removed, only the simplified complex $C(\Upsilon)$ remains.

**Lemma 6.** *The algebras associated to the 010-resolutions of $D_1$ and $D_2$ split as direct sums of $\mathcal{R}[x_0, \ldots, x_6, x_{13}, \ldots, x_n]$-modules $\mathcal{A}_{010}(D_i) \cong \mathcal{C}_{110}(D_i) \oplus \mathcal{C}_{011}(D_i)$, where $\mathcal{C}_{110}(D_i) \cong \mathcal{A}_{110}(D_i)$. The edge map $\mathcal{A}_{010}(D_i) \to \mathcal{A}_{110}(D_i)$ is an isomorphism when restricted to the first summand of $\mathcal{A}_{010}(D_i)$.*

**Lemma 7.** *The algebras associated to the 000-resolutions of $D_1$ and $D_2$ split as direct sums of $\mathcal{R}[x_0, \ldots, x_6, x_{13}, \ldots, x_n]$-modules $\mathcal{A}_{000}(D_i) \cong \mathcal{B} \oplus \mathcal{B}_{011}(D_i)$, where*

*$\mathcal{B}_{011}(D_i) \cong \mathcal{C}_{011}(D_i)$. The edge map $\mathcal{A}_{000} \to \mathcal{A}_{010}$ restricted to $\mathcal{B}_{011}(D_i)$ is an isomorphism onto $\mathcal{C}_{011}(D_i)$, the second summand of $\mathcal{A}_{010}$ in Lemma 6.*

*Proof of Lemma 6.* We know from Lemma 3 that $\mathcal{A}_{010}$ splits as a direct sum of modules isomorphic to $\mathcal{A}_{110}$ and $\mathcal{A}_{011}$. However, it will be useful to establish a particular splitting so that we may see directly the isomorphisms $\mathcal{C}_{110} \cong \mathcal{A}_{110}$ and (in the proof of Lemma 7) $\mathcal{C}_{011} \cong \mathcal{B}_{011}$.

The computations are similar to those used to prove Lemma 3. We first manipulate the local relations from the vertices in $D_i$ to a convenient form, then obtain direct sum splittings by eliminating all quadratic and higher-order appearances of one variable, and keep track throughout of how these manipulations affect the non-local relations.

We begin with the presentation of $\mathcal{A}_{010}$ as

$$\mathcal{A}_{010} \cong \frac{\mathcal{R}[x_0, \dots, x_n]}{\mathcal{L}_{123} + \mathcal{L} + \mathcal{N}_{010}},$$

where $\mathcal{L}_{123}$ is generated by local relations from layers $s_1$, $s_2$, and $s_3$, $\mathcal{L}$ is generated by the local relations associated to other layers, and $\mathcal{N}_{010}$ is generated by non-local relations. Note that $\mathcal{L}$ is generated by relations that do not use any of $x_7, \dots, x_{12}$. It will not be affected by any of the calculations below. Thinking of non-local relations as coming from coherent regions, notice that $x_7, \dots, x_{12}$ need not ever appear in a generating set for $\mathcal{N}_{010}$ because any coherent region containing the elementary region to the right of $x_7$ and $x_9$ can be assumed to include the bigon bounded by edges $x_7, x_8, x_9$, and $x_{10}$. The manipulations below will not affect such a generating set for $\mathcal{N}_{010}$.

The following matrix is a generating set for $\mathcal{L}_{123}$, with $x_{11}$ and $x_{12}$ already eliminated using linear relations $tx_3 - x_{11}$ and $tx_{12} - x_6$.

$$\begin{pmatrix} t(x_1+x_2) - (x_7+x_8) \\ t^2 x_1 x_2 - x_7 x_8 \\ t^3 x_3 - x_6 \\ t x_7 - x_9 \\ t(x_9 + x_{10}) - (x_4 + x_5) \\ (t x_9 - x_5)(x_4 - t x_9) \\ t x_8 - x_{10} \end{pmatrix}$$

Use row IV to eliminate $x_7$, row VII to eliminate $x_8$, and row V to eliminate $x_{10}$, then rearrange.

$$\begin{pmatrix} t(x_1+x_2) - t^{-2}(x_4+x_5) \\ t^2 x_1 x_2 + t^{-2} x_9^2 - t^{-3} x_9 (x_4 + x_5) \\ t^3 x_3 - x_6 \\ (t x_9 - x_5)(x_4 - t x_9) \end{pmatrix}$$

$$\Big\downarrow \text{I}+t^{-2}\text{III and II}+t^{-4}\text{IV}$$

$$\begin{pmatrix} t(x_1+x_2+x_3) - t^{-2}(x_4+x_5+x_6) \\ t^2 x_1 x_2 - t^{-4} x_4 x_5 \\ t^3 x_3 - x_6 \\ (t x_9 - x_5)(x_4 - t x_9) \end{pmatrix}$$

Clear negative powers of $t$ from all rows and symmetrize the presentation as follows.

$$\begin{pmatrix} t^3(x_1+x_2+x_3)-(x_4+x_5+x_6) \\ t^6x_1x_2-x_4x_5 \\ t^3x_3-x_6 \\ (tx_9-x_5)(x_4-tx_9) \end{pmatrix}$$
$$\Big\downarrow{\scriptstyle \mathrm{II}+t^3(x_1+x_2)\mathrm{III}+x_6\mathrm{I}-x_6\mathrm{III}}$$
$$\begin{pmatrix} t^3(x_1+x_2+x_3)-(x_4+x_5+x_6) \\ t^6\sigma_2(x_1,x_2,x_3)-\sigma_2(x_4,x_5,x_6) \\ t^3x_3-x_6 \\ (tx_9-x_5)(x_4-tx_9) \end{pmatrix}$$

where $\sigma_2$ is the second elementary symmetric polynomial.

Let $\overline{\mathcal{L}}_{123}$ denote the ideal generated by the first two rows above and $q=(tx_9-x_5)(x_4-tx_9)$. Notice that $\overline{\mathcal{L}}_{123}$ is generated by relations that do not use any of $x_7,\dots,x_{12}$. Define

$$\mathcal{T}=\frac{\mathcal{R}[x_0,\dots,x_6,x_{13},\dots,x_n]}{\mathcal{L}+\overline{\mathcal{L}}_{123}}.$$

So far, we have established that

$$\mathcal{A}_{010}\cong\frac{\mathcal{T}[x_9]}{(q)+(t^3x_3-x_6)+\mathcal{N}_{010}}$$

and that $x_9$ appears only in $q$. Since $q$ is quadratic in $x_9$, we could use it to replace any appearance of $x_9^k$ for $k\geq 2$ in a presentation of $\mathcal{A}_{010}$ with some polynomial that was linear in $x_9$. However, we have already eliminated all appearances of $x_9$ from the rest of the presentation. Therefore, we may forget the relation $q$, and split $\mathcal{A}_{010}$ into a summand generated by 1 and a summand generated by a polynomial that is linear in $x_9$.

$$\mathcal{A}_{010}\cong\frac{\mathcal{T}(1)}{(t^3x_3-x_6)+\mathcal{N}_{010}}\bigoplus\frac{\mathcal{T}(tx_9-x_5)}{(t^3x_3-x_6)+\mathcal{N}_{010}}.$$

With the first summand as $\mathcal{C}_{110}$ and the second as $\mathcal{C}_{011}$, this is the splitting asserted in the statement of the lemma.

We now check that $\mathcal{A}_{110}\cong\mathcal{C}_{110}=\frac{\mathcal{T}(1)}{(t^3x_3-x_6)+\mathcal{N}_{010}}$ by simplifying the presentation of $\mathcal{A}_{110}$. First note that $x_7,\dots,x_{12}$ do not appear in any local relations associated to layers $s_i$ for $i>3$. They also need not appear in a minimal generating set for $\mathcal{N}_{110}$. If a subset had one of these as an outgoing or incoming edge, we could add or remove bivalent vertices in layers $s_1$ and $s_2$ as necessary to eliminate them from the associated relation. Turning to $\mathcal{L}_{123}$, eliminate $x_{11}$ and $x_{12}$ immediately using

the linear relations on the rightmost strand, then remove $x_7, \ldots, x_{10}$ as follows.

$$\begin{pmatrix} tx_1 - x_8 \\ tx_2 - x_7 \\ t^3x_3 - x_6 \\ tx_7 - x_9 \\ tx_8 - x_{10} \\ t(x_9 + x_{10}) - (x_4 + x_5) \\ (tx_9 - x_4)(x_5 - tx_9) \end{pmatrix} \Big\downarrow{\scriptstyle \mathrm{I}+\mathrm{II}+t^{-2}\mathrm{III}+t^{-1}\mathrm{IV}+t^{-1}\mathrm{V}+t^{-2}\mathrm{VI}} \begin{pmatrix} t(x_1 + x_2 + x_3) - t^{-2}(x_4 + x_5 + x_6) \\ tx_2 - x_7 \\ t^3x_3 - x_6 \\ tx_7 - x_9 \\ tx_8 - x_{10} \\ t(x_9 + x_{10}) - (x_4 + x_5) \\ (tx_9 - x_4)(x_5 - tx_9) \end{pmatrix}$$

Simplify by multiplying the first row by $t^2$, using row II to eliminate $x_7$, using row V to eliminate $x_8$ and using row VI to eliminate $x_{10}$.

$$\begin{pmatrix} t^3(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6) \\ t^3x_3 - x_6 \\ t^2x_2 - x_9 \\ (tx_9 - x_4)(x_5 - tx_9) \end{pmatrix}$$

Now use row III to eliminate $x_9$.

$$\begin{pmatrix} t^3(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6) \\ t^3x_3 - x_6 \\ (t^3x_2 - x_4)(x_5 - t^3x_2) \end{pmatrix} \Big\downarrow{\scriptstyle \mathrm{III}+t^3(x_2+x_3)\mathrm{I}+(x_4+x_5-t^3(x_2+x_3))\mathrm{II}} \begin{pmatrix} t^3(x_1 + x_2 + x_3) - (x_4 + x_5 + x_6) \\ t^3x_3 - x_6 \\ t^6\sigma_2(x_1, x_2, x_3) - \sigma_2(x_4, x_5, x_6) \end{pmatrix}$$

The top and bottom rows are the generators of $\overline{\mathcal{L}}_{123}$. Therefore, we have

$$\mathcal{A}_{110} \cong \frac{\mathcal{T}}{(t^3x_3 - x_6) + \mathcal{N}_{110}}.$$

It remains to check that $\mathcal{N}_{110} = \mathcal{N}_{010}$. Figure 11 shows how the cycles that pass through the 010-resolution of $D_1$ pair up with the cycles that pass through the 110-resolution of $D_1$ to give equivalent non-local relations. Any cycle that does not pass through this region certainly has the same associated non-local relation in $\mathcal{N}_{010}$ and $\mathcal{N}_{110}$. We have identified identical generating sets for $\mathcal{N}_{010}$ and $\mathcal{N}_{110}$. Therefore, $\mathcal{A}_{110}$ and $\mathcal{C}_{110}$ have identical presentations. Since the edge map from $\mathcal{A}_{010}$ to $\mathcal{A}_{110}$ is defined to be the quotient map, it is an isomorphism when restricted to $\mathcal{C}_{110}$. $\square$

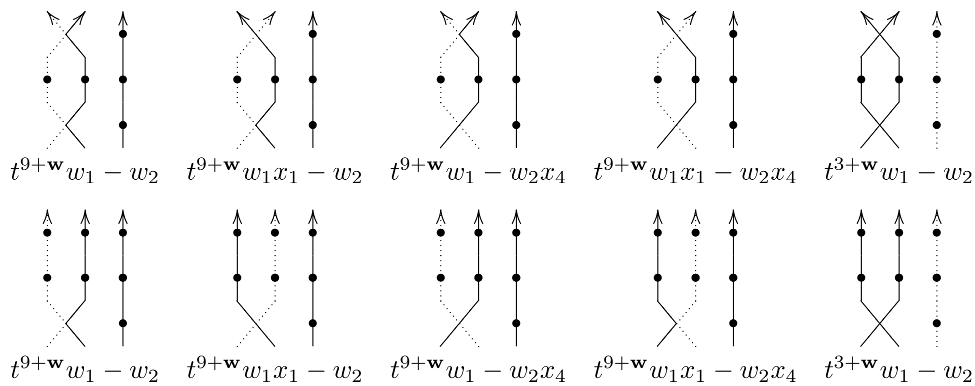



FIGURE 11. Pairing of cycles that pass through the 010-resolution (top row) and the 110-resolution (bottom row) of $D_1$. In each local picture, $\mathbf{w}$ is the weight, $w_1$ is the product of outgoing edges, and $w_2$ is the product of incoming edges for the portion of the cycle away from the portion of $D_1$ that is shown here.

The proof of Lemma 7 is similar, except that more work is required to keep track of the non-local relations. As before, we use local relations associated to layers $s_1$, $s_2$, and $s_3$ in $\mathcal{A}_{000}$ to eliminate several edge variables, then use a quadratic relation to split $\mathcal{A}_{000}$ as a direct sum, and finally check that one of the direct summands is in fact isomorphic to $\mathcal{C}_{011}$.

*Proof of Lemma 7.* Let $\mathcal{L}_{123}$ denote the ideal generated by local relations associated to layers $s_1$, $s_2$, and $s_3$, while $\mathcal{L}$ denotes the ideal generated by local relations associated to all other layers. As before, $\mathcal{N}_{000}$ will denote the ideal generated by non-local relations in the 000-resolution. So we begin with

$$\mathcal{A}_{000} \cong \frac{\mathcal{R}[x_0, \dots, x_n]}{\mathcal{L} + \mathcal{L}_{123} + \mathcal{N}_{000}}.$$

The general strategy will be to eliminate $x_7, \dots, x_{12}$ from the presentation of $\mathcal{A}_{000}$ and limit use of $x_9$ as much as possible. We will then rewrite $\mathcal{A}_{000}$ in the form $\mathcal{T}[x_9]/\mathcal{I}$ for an appropriate ideal $\mathcal{I}$, where $\mathcal{T}$ is the same algebra defined in the proof of Lemma 7. Finally, we will use the quadratic relation associated to layer $s_3$, which is $(tx_9 - x_4)(x_5 - tx_9)$, to split $\mathcal{A}_{000}$ into direct summands generated by 1 and $t^2x_3 - x_9$.

Notice first that no part of this strategy will affect the ideal $\mathcal{L}$. Edges $x_7, \dots, x_{12}$ connect layer $s_1$ to layer $s_2$ or layer $s_2$ to layer $s_3$, so they do not appear in local relations associated to any other layers.

For the relations in $\mathcal{L}_{123}$, first use the relations $tx_3 - x_{11}$ and $tx_{12} - x_6$ to replace $x_{11}$ and $x_{12}$. Then the following matrix is a generating set for $\mathcal{L}_{123}$ in the 000-resolution of $D_1$.

$$\begin{pmatrix} t(x_1+x_2)-(x_7+x_8) \\ (tx_2-x_7)(x_8-tx_2) \\ t(tx_3+x_7)-(t^{-1}x_6+x_9) \\ (t^3x_3-x_6)(t^{-1}x_9-tx_3) \\ t(x_9+x_{10})-(x_4+x_5) \\ (tx_9-x_5)(tx_9-x_4) \\ tx_8-x_{10} \end{pmatrix}$$

$$\Big\downarrow{\scriptstyle \mathrm{I}+t^{-1}\mathrm{III}+t^{-2}\mathrm{V}+t^{-1}\mathrm{VII}}$$

$$\begin{pmatrix} t(x_1+x_2+x_3)-t^{-2}(x_4+x_5+x_6) \\ (tx_2-x_7)(x_8-tx_2) \\ t(tx_3+x_7)-(t^{-1}x_6+x_9) \\ (t^3x_3-x_6)(t^{-1}x_9-tx_3) \\ t(x_9+x_{10})-(x_4+x_5) \\ (tx_9-x_5)(tx_9-x_4) \\ tx_8-x_{10} \end{pmatrix}$$

Next, multiply row I by $t^2$, use row III to eliminate $x_7$, use row V to eliminate $x_{10}$, and use row VII to eliminate $x_8$, then multiply row II by $t^4$.

$$\begin{pmatrix} t^3(x_1+x_2+x_3)-(x_4+x_5+x_6) \\ (t^3(x_2+x_3)-x_6-tx_9)(x_4+x_5-tx_9-t^3x_2) \\ (t^3x_3-x_6)(t^{-1}x_9-tx_3) \\ (tx_9-x_5)(tx_9-x_4) \end{pmatrix}$$

Use row IV to replace $t^2x_9^2$ in row II, then add $t^2$III and $t^3(x_2+x_3)$I to row II, and then multiply row III by $t$ to obtain

$$\begin{pmatrix} t^3(x_1+x_2+x_3)-(x_4+x_5+x_6) \\ t^6\sigma_2(x_1,x_2,x_3)-\sigma_2(x_4,x_5,x_6) \\ (t^3x_3-x_6)(x_9-t^2x_3) \\ (tx_9-x_5)(tx_9-x_4) \end{pmatrix}$$

where $\sigma_2$ is the second elementary symmetric polynomial. As in the proof of Lemma 6, let $\overline{\mathcal{L}}_{123}$ denote the ideal generated by the relations in rows I and II, and recall that $\overline{\mathcal{L}}_{123}$ is generated by relations that do not use $x_9$. Let $p = (t^3x_3 - x_6)(x_9 - t^2x_3)$ and $q = (tx_9 - x_5)(tx_9 - x_4)$. Then we have reduced the original generating set for $\mathcal{L}_{123}$ to a generating set that does not involve $x_9$, the quadratic relation $q$ that will be used to split $\mathcal{A}_{000}$ as a direct sum, and the relation $p$, which we will have to follow up carefully. So far, we have

$$\mathcal{A}_{000} \cong \frac{\mathcal{T}[x_7,\ldots,x_{12}]}{(p)+(q)+\mathcal{N}_{000}}.$$

Using the coherent regions definition for the generators of $\mathcal{N}_{000}$, we can split $\mathcal{N}_{000}$ into a sum of five ideals based on types of coherent regions. Label the elementary regions in the vicinity of the Reidemeister III move as in Figure 12. As usual, assume that the braid axis is to the right of the diagram. Let $\mathcal{N}$ be the ideal generated by the relations from coherent regions that do not use any of $E_1$,$E_2$,$E_3$,

or $E_4$. None of these relations use edge variables $x_7,\ldots,x_{12}$, so they will carry through all of our calculations unchanged.

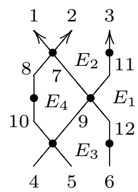



FIGURE 12. Elementary regions in the vicinity of the Reidemeister III move in the 000-resolution of $D_1$.

Let $\mathcal{E}_{1234}$ be generated by relations from coherent regions that use all of $E_1$,$E_2$,$E_3$, and $E_4$. These relations use $x_1$ and $x_4$, but not any of $x_7,\ldots,x_{12}$, so they carry through our calculations unchanged. The ideal $\mathcal{E}_{1234}$ also accounts for relations associated to coherent regions that contain $E_1$,$E_2$, and $E_3$. Adding $E_4$ to such a region would add only the bivalent vertex between edges 8 and 10, which is exactly the situation described in Observation 2. Therefore, we need not consider coherent regions that contain $E_1$,$E_2$, and $E_3$ without $E_4$ in a minimal generating set for $\mathcal{N}_{000}$.

Let $\mathcal{E}_{12}$ (respectively $\mathcal{E}_{13}$) be generated by non-local relations from coherent regions that use $E_1$ and $E_2$, but not $E_3$ or $E_4$ (respectively $E_1$ and $E_3$ but not $E_2$ or $E_4$). Some of the edge variables $x_7,\ldots,x_{12}$ do appear in the relations associated to such regions, but can be easily eliminated using the quadratic relations from layers $s_1$ or $s_3$ as appropriate. Figure 13 shows the necessary calculations in each case.

Finally, let $\mathcal{E}_1$ be generated by relations from coherent regions that use $E_1$ but none of $E_2, E_3$, or $E_4$, as shown in Figure 14. These relations have the form $t^{4+\mathbf{w}}w_{\text{out}}x_7 - w_{\text{in}}x_9$, where $\mathbf{w}$, $w_{\text{out}}$, and $w_{\text{in}}$ come from pieces of the coherent region not shown in Figure 14. We will not be able to simultaneously eliminate $x_7,\ldots,x_{12}$ from these relations, but we can eliminate all but $x_9$ using the linear relations from the crossing in layer $s_2$ and linear relations associated to bivalent vertices. In fact, we can write any generator of $\mathcal{E}_1$ in the form $t^{2+\mathbf{w}}w_{\text{out}}(x_6 - t^3x_3) + x_9(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}})$, where $w_{\text{out}}$ and $w_{\text{in}}$ are words in $x_0,\ldots,x_6,x_{13},\ldots,x_n$ only.

We have exhausted the possible combinations of elementary regions $E_1,\ldots,E_4$ that can appear in a coherent region, so we may now express $\mathcal{N}_{000}$ as

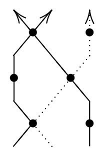
$$\mathcal{N}_{000} = \mathcal{N} + \mathcal{E}_{1234} + \mathcal{E}_{12} + \mathcal{E}_{13} + \mathcal{E}_1.$$

Moreover, we have eliminated all appearances of $x_7,\ldots,x_{12}$ from the generating sets of $\mathcal{N}$, $\mathcal{E}_{1234}$, $\mathcal{E}_{12}$, and $\mathcal{E}_{13}$. Defining $\mathcal{T}'$ by

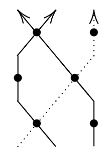
$$\frac{\mathcal{T}}{\mathcal{E}_{12}+\mathcal{E}_{13}+\mathcal{E}_{1234}+\mathcal{N}}$$

we then have a presentation of $\mathcal{A}_{000}$ as

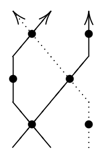
$$\mathcal{A}_{000} \cong \frac{\mathcal{T}'[x_9]}{(p)+(q)+\mathcal{E}_1}.$$



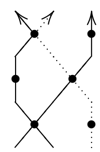
$$t^{6+\mathbf{w}} w_{\text{out}} x_7 x_{10} - w_{\text{in}} x_4$$
$$\equiv t^{9+\mathbf{w}} w_{\text{out}} x_1 x_2 - w_{\text{in}} x_4$$



$$t^{6+\mathbf{w}} w_{\text{out}} x_7 x_{10} - w_{\text{in}}$$
$$\equiv t^{9+\mathbf{w}} w_{\text{out}} x_1 x_2 - w_{\text{in}}$$

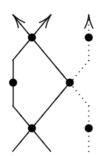


$$t^{6+\mathbf{w}} w_{\text{out}} - w_{\text{in}} x_8 x_9$$
$$\equiv t^{9+\mathbf{w}} w_{\text{out}} - w_{\text{in}} x_4 x_5$$



$$t^{6+\mathbf{w}} w_{\text{out}} x_1 - w_{\text{in}} x_8 x_9$$
$$\equiv t^{9+\mathbf{w}} w_{\text{out}} x_1 - w_{\text{in}} x_4 x_5$$

FIGURE 13. Relations that generate $\mathcal{E}_{12}$ and $\mathcal{E}_{13}$, along with modifications to avoid the use of $x_7, \ldots, x_{12}$. In each diagram, $\mathbf{w}$, $w_{\text{out}}$, and $w_{\text{in}}$ come from the portions of the cycle not shown in these local pictures. Assume the braid axis is to the right in each diagram.



$$\begin{aligned} &t^{4+\mathbf{w}} w_{\text{out}} x_7 - w_{\text{in}} x_9 \\ &\equiv t^{4+\mathbf{w}} w_{\text{out}} \left(t^{-2} x_6 + t^{-1} x_9 - t x_3\right) - w_{\text{in}} x_9 \\ &\equiv t^{2+\mathbf{w}} w_{\text{out}} \left(x_6 - t^3 x_3\right) + x_9 \left(t^{3+\mathbf{w}} w_{\text{out}} - w_{\text{in}}\right) \end{aligned}$$

FIGURE 14. Removing $x_7$ from relations that generate $\mathcal{E}_1$.

The next step will be to use $q$ to split $\mathcal{A}_{000}$ as a direct sum of $\mathcal{R}$-modules, one of which is generated by 1 and one of which is generated by $t^2 x_3 - x_9$. In other words, we would like to find ideals $\mathcal{P}^1, \mathcal{P}^x, \mathcal{E}_1^1$, and $\mathcal{E}_1^x$ in $\mathcal{T}'$ such that
$$\frac{\mathcal{T}'[x_9]}{(p) + (q) + \mathcal{E}_1} \cong \frac{\mathcal{T}'(1)}{\mathcal{P}^1 + \mathcal{E}_1^1} \oplus \frac{\mathcal{T}'(t^2 x_3 - x_9)}{\mathcal{P}^x + \mathcal{E}_1^x}$$
as $\mathcal{R}$-modules.

As in the proof of Lemma 6, we may use $q$ to replace any appearance of $x_9^k$ for $k \geq 2$ with a polynomial that is linear in $x_9$. This procedure has no effect on the ideals from which $x_9$ has been eliminated, but it does affect $(p)$ and $\mathcal{E}_1$. To

analyze how, think of the ideal that $p$ generates in $\mathcal{T}'[x_9]/(q)$ as the sum of the ideals generated by $p$ and $x_9p$. If we use $q$ to eliminate any appearances of $x_9^2$ in these generating sets, then we can find appropriate generators for $\mathcal{P}^1$ and $\mathcal{P}^x$ by writing $p$ and $x_9p$ in terms of 1 and $t^2x_3 - x_9$. Actually, $p = (t^3x_3 - x_6)(x_9 - t^2x_3)$ is already in the correct format, so let $t^3x_3 - x_6$ be one of the generators of $\mathcal{P}^x$. For $x_9p$, we first calculate $x_9(x_9 - t^2x_3)$, replacing $x_9$ using $q$, then eliminating a term using $p$.

$$\begin{aligned}
&x_9(x_9 - t^2x_3)\\
&= t^{-1}x_9x_4 + t^{-1}x_9x_5 - t^{-2}x_4x_5 - t^2x_3x_9\\
&= (x_9 - t^2x_3)(t^{-1}x_4 + t^{-1}x_5 - t^2x_3) - t^{-2}x_4x_5 - t^4x_3^2 + tx_3x_4 + tx_3x_5\\
&\equiv t^3x_3x_4 + t^3x_3x_5 - t^6x_3^2 - x_4x_5\\
&= (t^3x_3 - x_4)(x_5 - t^3x_3) \qquad (18)
\end{aligned}$$

Therefore, the ideal generated by $x_9p$ in $\mathcal{S}[x_9]/(q)$ is equal to the ideal generated by $(t^3x_3 - x_4)(x_5 - t^3x_3)(t^3x_3 - x_6)$, which no longer uses $x_9$. Let $\mathcal{P}^1$ be the ideal generated by this relation in $\mathcal{T}'$. Adding this generator to $\mathcal{P}^x$ would not change the ideal, since $\mathcal{P}^x$ already has $t^3x_3 - x_6$ as a generator.

We use the same strategy to find appropriate generators for $\mathcal{E}_1^1$ and $\mathcal{E}_1^x$. Generators of $\mathcal{E}_1$ have the form $f = t^{\mathbf{w}+4}w_{\text{out}}x_7 - w_{\text{in}}x_9$. We would like to write $f$ and $x_9f$ in terms of 1 and $t^2x_3 - x_9$. We have already seen that

$$f \equiv t^{2+\mathbf{w}}w_{\text{out}}\left(x_6 - t^3x_3\right) + x_9\left(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}\right),$$

where $w_{\text{out}}$ and $w_{\text{in}}$ are words in $x_0, \ldots, x_6, x_{13}, \ldots, x_n$ only. Factoring out $x_9 - t^2x_3$ yields

$$f \equiv (x_9 - t^2x_3)(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}) + t^2(t^{\mathbf{w}}w_{\text{out}}x_6 - w_{\text{in}}x_3).$$

Conveniently, the second term is a multiple of a generator of $\mathcal{N}$ obtained as follows. Suppose $f$ came from a coherent region $R$. Let $V_R$ be the set of vertices contained in the closure of $R$, so that $f$ is the relation associated to $V_R$ under the subset interpretation of the non-local relations. Delete from $V_R$ the 4-valent vertex in layer $s_2$, the bivalent vertex between edges 3 and 11, and the bivalent vertex between edges 12 and 6. These deletions drop the weight of $V_R$ by 4. The resulting set of vertices has the same incoming and outgoing edges as $V_R$ except that $x_7$ has been replaced by $x_6$ and $x_9$ has been replaced by $x_3$. Therefore, the relation associated to this subset, which must appear in $\mathcal{N}$, is exactly $t^{\mathbf{w}}w_{\text{out}}x_6 - w_{\text{in}}x_3$. The above expression for $f$ then simplifies to

$$(x_9 - t^2x_3)(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}). \qquad (19)$$

We conclude that a generating set for $\mathcal{E}_1^x$ should include $t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}$.

Next consider $x_9f$, using the final expression for $f$ obtained in Equation 19 and the expression for $x_9(x_9 - t^2x_3)$ obtained in Equation 18.

$$\begin{aligned}
x_9f &= x_9(x_9 - t^2x_3)(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}})\\
&\equiv (t^3x_3 - x_4)(x_5 - t^3x_3)(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}) \qquad (20)
\end{aligned}$$

These calculations eliminate all appearances of $x_7, \ldots, x_{12}$ from $x_9f$. Since we have already put $t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}}$ in the generating set of $\mathcal{E}_1^x$, Equation 20 is automatically included. Let $\mathcal{E}_1^1$ be the ideal generated in $\mathcal{T}$ by $(t^3x_3 - x_4)(x_5 - t^3x_3)(t^{3+\mathbf{w}}w_{\text{out}} - w_{\text{in}})$.

We have now split $\mathcal{A}_{000}$ as a direct sum of $\mathcal{R}$-modules:

$$\mathcal{A}_{000} \cong \frac{\mathcal{T}'(1)}{\mathcal{P}^1 + \mathcal{E}_1^1} \oplus \frac{\mathcal{T}'(t^2x_3 - x_9)}{\mathcal{P}^x + \mathcal{E}_1^x}.$$

Define $\mathcal{B}$ to be the first summand and $\mathcal{B}_{011}$ to be the second.

It remains only to check that $\mathcal{B}_{011} \cong \mathcal{C}_{011}$. So far, we have a presentation of $\mathcal{B}_{011}$ as

$$\mathcal{B}_{011} \cong \frac{\mathcal{T}}{\mathcal{E}_{12} + \mathcal{E}_{13} + \mathcal{E}_{1234} + \mathcal{N} + \mathcal{P}_1^x + \mathcal{E}_1^x}.$$

The proof of Lemma 6 gave a presentation of $\mathcal{C}_{011}$ as

$$\mathcal{C}_{011} \cong \frac{\mathcal{T}}{(t^3x_3 - x_6) + \mathcal{N}_{010}}.$$

By definition, $\mathcal{P}^x = (t^3x_3 - x_6)$, so the work is entirely in checking that the non-local relations in the 010-resolution are the same as those in $\mathcal{B}_{011}$.

Use the coherent regions definition of the non-local relations with elementary regions labeled as in Figure 15 to classify the generators of $\mathcal{N}_{010}$. Coherent regions that do not use any of $F_1, F_2$, or $F_3$ match one to one with coherent regions in the 000-resolution that do not use any of $E_1, E_2, E_3$, or $E_4$ and give the same non-local relations. By Observation 2, coherent regions that use $F_1$ and $F_2$ may as well use $F_3$. These match one to one with the regions that define $\mathcal{E}_{1234}$ and give the same relations. Finally, the coherent regions that use only $F_1$ give generators for $\mathcal{N}_{010}$ with exactly the form of generators for $\mathcal{E}_1^x$. Therefore, $\mathcal{N}_{010} = \mathcal{E}_{1234} + \mathcal{N} + \mathcal{E}_1^x$.

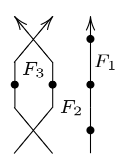



FIGURE 15. Elementary regions in the 010-resolution of $D_1$.

The remaining ideals $\mathcal{E}_{12}$ and $\mathcal{E}_{13}$ were necessary to generate $\mathcal{N}_{000}$ but are in fact redundant in the summand $\mathcal{B}_{011}$. The relations from Figure 13 used to define $\mathcal{E}_{12}$ and $\mathcal{E}_{13}$ correspond to subsets in the 010-resolution as follows. Suppose $R \cup E_1 \cup E_i$ is a coherent region for a generator of $\mathcal{E}_{12}$ or $\mathcal{E}_{13}$. Let $V_{R\cup E_1 \cup E_i}$ be the corresponding subset in the 000-resolution. Let $V'_{R\cup E_1\cup E_i}$ be the same subset of vertices in the 010-resolution, but with the 4-valent vertex in layer $s_2$ replaced by the two bivalent vertices created by resolving it. Then $V'_{R\cup E_1\cup E_i}$ yields the same relation in $\mathcal{N}_{010}$ as $R \cup E_1 \cup E_i$ did in $\mathcal{N}_{000}$. Figure 16 shows what these subsets look like in the vicinity of the Reidemeister move and how they correspond to the appropriate relations.

Having completed the verification that $\mathcal{N}_{010} = \mathcal{E}_{1234} + \mathcal{N} + \mathcal{E}_1^x + \mathcal{E}_{12} + \mathcal{E}_{13}$, we have now showed that $\mathcal{C}_{011}$ and $\mathcal{B}_{011}$ have identical presentations. Since the edge map $\mathcal{A}_{000} \to \mathcal{A}_{010}$ is by definition the quotient map, it is an isomorphism when restricted to $\mathcal{B}_{011} \to \mathcal{C}_{011}$. $\square$

$t^{6+\mathbf{w}}w_{\mathrm{out}} - w_{\mathrm{in}}x_8x_9$ or $t^{6+\mathbf{w}}w_{\mathrm{out}}x_1 - w_{\mathrm{in}}x_8x_9$

$t^{6+w}w_{\mathrm{out}}x_7x_{10} - w_{\mathrm{in}}$ or $t^{6+w}w_{\mathrm{out}}x_7x_{10} - w_{\mathrm{in}}x_4$

FIGURE 16. Subsets (shown with open circles) in the 010-resolution of $D_1$ that yield the relations corresponding to coherent regions in the 000-resolution that generate $\mathcal{E}_{12}$ (left) and $\mathcal{E}_{13}$ (right). As usual, $\mathbf{w}$, $w_{\mathrm{in}}$, and $w_{\mathrm{out}}$ refer to the portion of the subset not shown in these local diagrams.

## 8. STABILIZATION / REIDEMEISTER MOVE I

Let $D$ and $D^+$ (respectively $D^-$) be knot projections that differ by a positive (respectively negative) stabilization with labels as shown in Figure 17. In this section, we prove that $C(D^+)$ and $C(D^-)$ are chain homotopy equivalent to $C(D)$ as complexes of $\widehat{\mathcal{R}}[\underline{x}(D)]$-modules. The proof presented here requires the completion of the ground ring because we invert an element of the form $1-t^k$, but it is interesting to note that invariance under all of the other Markov moves holds over $\mathcal{R}$.

Since we have already established invariance under conjugation, we may assume that the stabilization occurs in layer $s_0$. By Section 7, we may assume it occurs on the outermost strand. As shown in Figure 18, any resolution in which the crossing in layer $s_0$ is smoothed is disconnected, so by Observation 4, the associated algebra will vanish. Therefore, it suffices to show that the algebra associated to the $I$-resolution of $D$ is isomorphic to the algebra associated to the corresponding resolution of $D^+$ or $D^-$ in which $s_0$ is singular.

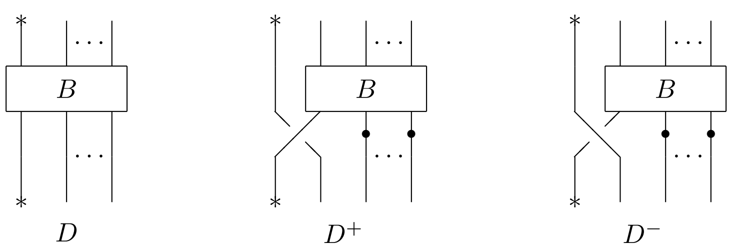



FIGURE 17. Diagrams $D$, $D^+$, and $D^-$. Assume the braid axis is to the right of each picture and all strands are oriented upwards.

We proceed via an intermediary diagram $D^\bullet$ shown on the right in Figure 18. To go from $D^\bullet$ back to $D$, first remove the layer of bivalent vertices just above the basepoint. By Proposition 2, this transforms $\mathcal{A}_J(D^\bullet)$ to $\mathcal{A}_J(D^\bullet)\otimes_{(\mathcal{R},\psi)}\mathcal{R}$. Use the linear relation $tx_n - x_1$ to remove $x_1$ from the presentation of $\mathcal{A}_J(D^\bullet)\otimes_{(\mathcal{R},\psi)}\mathcal{R}$ without changing its isomorphism type. This process leaves exactly diagram $D$.

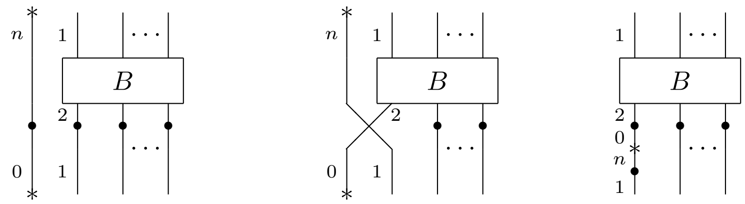



FIGURE 18. From left to right: the smoothed resolution of layer $s_0$ in $D^+$ or $D^-$; the singular resolution of crossing $s_0$ in $D^+$ or $D^-$; the diagram $D^\bullet$. Assume the braid axis is to the right of each diagram and all strands are oriented upwards.

**Lemma 8.** *Let $D^+$ and $D^-$ be the diagrams shown in Figure 17 and $D^\bullet$ be the diagram on the right in Figure 18. Then for any multi-index $J$, there are isomorphisms of $\widehat{\mathcal{R}}[\underline{x}(D^\bullet)]$-modules (equivalently $\widehat{\mathcal{R}}[\underline{x}(D^+)]$- or $\widehat{\mathcal{R}}[\underline{x}(D^-)]$-modules)*

$$\begin{aligned}\mathcal{A}_J(D^\bullet) &\cong \mathcal{A}_{0J}(D^+) \quad \textit{and}\\ \mathcal{A}_J(D^\bullet) &\cong \mathcal{A}_{1J}(D^-).\end{aligned}$$

*Proof.* Since the $0J$-resolution of $D^+$ and the $1J$-resolution of $D^-$ are identical as diagrams, we will refer to $D^+$ throughout without loss of generality. The following matrix contains a generating set for $\mathcal{L}_0$, the non-local relation associated to the outermost cycle (see Observation 3), and the non-local relation associated to the set of all vertices except the 4-valent vertex in layer $s_0$ in $\mathcal{A}_{0J}(D^+)$.

$$\begin{pmatrix} t(x_n+x_2)-(x_0+x_1) \\ t^2x_nx_2 - x_0x_1 \\ t^{\mathbf{w}(D)+b+1}x_n - x_0 \\ t^{\mathbf{w}(D)+b-1}x_1 - x_2 \end{pmatrix} \xrightarrow[\mathrm{II}+t^2x_n\mathrm{IV}]{\mathrm{I}-\mathrm{III}+t\mathrm{IV}} \begin{pmatrix} (1-t^{\mathbf{w}(D)+b})(tx_n - x_1) \\ x_1(t^{\mathbf{w}(d)+b+1}x_n - x_0) \\ t^{\mathbf{w}(D)+b+1}x_n - x_0 \\ t^{\mathbf{w}(D)+b-1}x_1 - x_2 \end{pmatrix}$$

Since $1-t^{\mathbf{w}(D)+b}$ is a unit in $\widehat{\mathcal{R}}$, we can eliminate that factor from the top row. We can eliminate row II because it is a multiple of row III. A bit more simplification leaves exactly the linear relations associated with the arrangement of bivalent vertices and the basepoint on the outermost strand of $D^\bullet$.

$$\begin{pmatrix} tx_n - x_1 \\ t^{\mathbf{w}(D)+b+1}x_n - x_0 \\ t^{\mathbf{w}(D)+b-1}x_1 - x_2 \end{pmatrix} \xrightarrow[\text{mult. III by } -t]{\mathrm{III}+t^{\mathbf{w}(D)+b-1}\mathrm{I}-t^{-1}\mathrm{II}} \begin{pmatrix} tx_n - x_1 \\ t^{\mathbf{w}(D)+b+1}x_n - x_0 \\ tx_2 - x_0 \end{pmatrix}$$

All other local relations in diagram $D^\bullet$ are exactly the same as those in the $0J$-resolution of $D^+$. For any subset in the $0J$-resolution of $D^+$ that does not include the 4-valent vertex in layer $s_0$, the corresponding subset in the $J$-resolution of $D^\bullet$ has the same associated non-local relation. Any subset in the $0J$-resolution of $D^+$ that does include the 4-valent vertex in layer $s_0$ has a corresponding subset in the $J$-resolution of $D^\bullet$ given by the two bivalent vertices nearest to the basepoint and an appropriate subset in the rest of the diagram. These also give the same non-local relation. Therefore, $\mathcal{N}_{0J} \subset \mathcal{N}_J$. By Observation 1, adjacent bivalent vertices can always be added or removed from a subset without changing the associated non-local relation, so $\mathcal{N}_J \subset \mathcal{N}_{0J}$. This completes the verification that $\mathcal{A}_{0J}(D^+)$ and $\mathcal{A}_J(D^\bullet)$ have identical presentations over the completed edge ring $\widehat{\mathcal{R}}[\underline{x}(D^\bullet)]$. $\square$

## 9. Identification with knot Floer homology

The set-up of the cube of resolutions in Section 2 of this paper differs somewhat from Ozsváth and Szabó's original formulation [12], so it does not follow formally from their work that $C(D)$, as defined in (2) of this paper, computes knot Floer homology. However, an adaptation of the arguments in Sections 3–5 of [12], suffices to prove the following result, which is an analogue of [12, Theorem 1.2].

**Proposition 9.** *Let $D$ be a layered braid diagram with initial edge $x_0$. Then there is an isomorphism of graded $\mathbb{F}_2[x_0]$-modules*

$$H_*(C(D) \otimes_{\mathcal{R}[\underline{x}(D)]} \widehat{\mathcal{R}[\underline{x}(D)]} \otimes \mathbb{F}_2) \cong HFK^-(K) \otimes_{\mathbb{F}_2} \mathbb{F}_2[t^{-1}, t]]$$

*and an isomorphism of graded $\mathbb{F}_2$-vector spaces*

$$H_*(C(D)/(x_0) \otimes_{\mathcal{R}[\underline{x}(D)]} \widehat{\mathcal{R}[\underline{x}(D)]} \otimes \mathbb{F}_2) \cong \widehat{HFK}(K) \otimes_{\mathbb{F}_2} \mathbb{F}_2[t^{-1}, t]].$$

The two key differences between our set-up and that of [12] are the use of layered braid diagrams and the ground ring over which we define the cube of resolutions chain complex. Ozsváth and Szabó use a knot projection in braid form with a basepoint $*$, but do not require the additional bivalent vertices that we add parallel to each crossing when forming a layered braid diagram. Consequently, in their diagrams, bivalent vertices arise only when a crossing is smoothed, which means they come in pairs that lie on adjacent strands. A layered braid diagram has these sorts of bivalent vertices, but also others. This difference will require us to modify the Heegaard diagrams used in the proof of [12, Theorem 1.2].

The second difference between our set-up and that of [12] is in the ground rings over which the cube of resolutions complex is defined. We define the algebras $\mathcal{A}_I(D)$ over $\mathcal{R}[\underline{x}(D)] = \mathbb{Z}[t^{-1}, t][\underline{x}(D)]$, and pass to the completion $\widehat{\mathcal{R}} = \mathbb{Z}[t^{-1}, t]][\underline{x}(D)]$ for the precise statement of invariance in Section 2.4. Ozsváth and Szabó set up their algebras over $\mathbb{F}_2[\underline{x}(D), t]$, pass to the completion $\mathbb{F}_2[\underline{x}(D)][[t]]$ when identifying these algebras with twisted singular knot Floer homology and finally pass to $\mathbb{F}_2[\underline{x}(D)][t^{-1}, t]]$ for the statement of [12, Theorem 1.2]. These choices of rings in each case allow results to be stated in the greatest possible generality, but a profusion of tensor products will be required to bring the two approaches into alignment.

*Proof.* Ozsváth and Szabó prove [12, Theorem 1.2] in three steps: calculate a particular twisting of singular knot Floer homology to verify that it is identical to the algebra they define as a quotient of the edge ring [12, Section 3]; establish a spectral sequence from the cube of resolutions defined algebraically to knot Floer homology [12, Section 4]; show that the spectral sequence collapses [12, Section 5]. We mirror each of these arguments in turn, pointing out where modifications are required to address the differences between our set-up (Section 2 of this paper) and that of [12].

Let $S$ be a layered braid diagram with all crossings singularized or smoothed. The twisted version of singular knot Floer homology needed to recover the algebra $\mathcal{A}(S)$ as defined in (1) in Section 2.1 of this paper is specified by the "initial diagram" in [12, Figure 3] with the additional rule that near a bivalent vertex that does not arise from smoothing a crossing, the diagram has the form shown on the left in Figure 19. Near a pair of bivalent vertices that arise from smoothing a crossing, we use the same diagram as in [12, Figure 3], which is shown in the middle in Figure 19. Call this the *modified initial diagram.* Let $\underline{CFK}^-(S)$ denote the chain complex coming from the modified initial diagram. That is, $\underline{CFK}^-(S)$ is the $\mathbb{F}_2[\underline{x}(S)][[t]]$-module whose

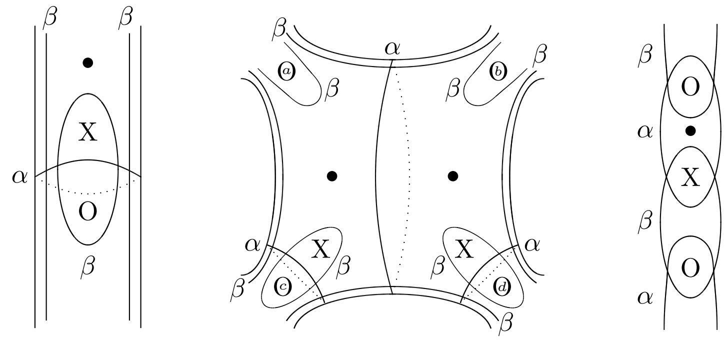



FIGURE 19. From left to right: the modified initial diagram near an extra bivalent vertex; the modified initial diagram near a bivalent vertex arising from a smoothing; the planar diagram or the master diagram near any bivalent vertex. The bold dots in each picture show the marking that specifies our particular twisted version of singular knot Floer homology.

generators are given by intersection points and differentials by counting holomorphic disks with respect to the twisting in the modified initial diagram. See [9] for a precise definition of singular knot Floer homology, [12, Section 2.1] for details on twisted coefficients in knot Floer homology generally, and [12, Section 3.1] for details on combining singular knot Floer homology with twisted coefficients. The completion of the ground ring with respect to $t$ is necessary to make the differential on twisted singular knot Floer homology well defined, as detailed in [12, Section 3.1]. We will continue to work over $\mathbb{F}_2[\underline{x}(S)][[t]]$ for the first section of this proof, so abbreviate this ring by $\mathcal{R}'$.

Let $M$ denote the Koszul complex on the linear relations for each vertex.

$$M = \bigotimes_{v \in V_4} \left( \mathcal{R}' \xrightarrow{tx_a^{(v)} + tx_b^{(v)} - x_c^{(v)} - x_d^{(v)}} \mathcal{R}' \right) \otimes \bigotimes_{v \in V_2} \left( \mathcal{R}' \xrightarrow{tx_a^{(v)} - x_c^{(v)}} \mathcal{R}' \right),$$

where $V_4$ and $V_2$ denote the set of 4-valent and bivalent vertices, respectively, in $S$. Let $C'(S) = \underline{CFK}^-(S) \otimes M$. Then the claim, an analogue of [12, Theorem 3.1], is that we can identify $H_*(C'(S))$ with $\mathcal{A}(S)$ after appropriately changing the ground rings. Recall that $\mathcal{A}(S)$ was defined in (1) of Section 2.1 of this paper as an $\mathcal{R}[\underline{x}(S)] = \mathbb{Z}[t^{-1}, t][\underline{x}(S)]$-module. Therefore, the precise claim is that

$$H_*(C'(S)) \otimes_{\mathcal{R}'} \mathcal{R}'[t^{-1}] \cong \mathcal{A}(S) \otimes_{\mathcal{R}[\underline{x}(S)]} \widehat{\mathcal{R}[\underline{x}(S)]} \otimes \mathbb{F}_2. \tag{21}$$

The reduced version of the statement,

$$H_*(C'(S)/(x_0)) \otimes_{\mathcal{R}'} \mathcal{R}'[t^{-1}] \cong \mathcal{A}(S)/(x_0) \otimes_{\mathcal{R}[\underline{x}(S)]} \widehat{\mathcal{R}[\underline{x}(S)]} \otimes \mathbb{F}_2, \tag{22}$$

then follows immediately.

The arguments required to prove [12, Proposition 3.4] apply essentially unchanged to show that $H_*(C'(S)/(x_0))$ is free as a $\mathbb{F}_2[[t]]$-module, generated by the generalized Kauffman states defined in [12, Figure 4], and concentrated in a single algebraic grading. The unreduced $H_*(C'(S))$ is also concentrated in a single algebraic grading. To calculate the structure of $H_*(C'(S))$ as an $\mathcal{R}'$-module, we

use a planar Heegaard diagram for $S$ defined exactly as in [12, Figure 9] with extra bivalent vertices of the layered diagram treated as if they had come from smoothing a crossing. So, the diagram looks like that on the right in Figure 19 near any bivalent vertex. The same procedure of handleslides and destabilizations described in the proof of [12, Lemma 3.7] shows that the chain complex specified by this planar diagram is quasi-isomorphic to the one specified by the modified initial diagram. The planar diagram has a canonical generator, which is a cycle, defined by making the same choice of intersection point near each vertex as Ozsváth and Szabó do in [12, Proposition 3.10]. Incoming differentials from chains with algebraic grading one higher than the canonical generator produce all of the quadratic local relations, the linear local relations associated to bivalent vertices, and the non-local relations that appear in the definition of $\mathcal{A}(S)$. Since $H_*(C'(S))$ is concentrated in a single algebraic grading, this completes the calculation and establishes the isomorphisms claimed in (21) and (22).

Now consider a layered braid diagram $D$ with $m$ crossings, and let $D_I$ denote its $I$-resolution. The spectral sequence constructed in [12, Section 4] comes from a filtration on

$$V(D) = \bigoplus_{I \in \{0,1\}^m} H_*\big(\underline{CFK}^-(D_I) \otimes M_I\big)\,,$$

where $M_I$ is the Koszul complex on linear relations coming from all vertices in diagram $D_I$. To define the filtration, Ozsváth and Szabó consider a planar Heegaard diagram that simultaneously encodes each possible state (positive, negative, singularized, smoothed) of a crossing [12, Figure 12]. To adapt this Heegaard diagram to $D$, we need only add a small piece like that shown on the right in Figure 19 near any bivalent vertex. Call the diagram from [12, Figure 12] so adapted the *master diagram*.

Using particular choices of generators near crossings in the master diagram, Ozsváth and Szabó define a filtration on $V(D)$. They also define maps that count holomorphic disks intersecting certain regions near crossings in the master diagram [12, Section 4]. In [12, Proposition 5.2], they verify that some of these maps (those with the appropriate gradings) are the same as the edge maps in Section 2.2 of this paper, under the identification of $H_*(C'(D_I))$ with $\mathcal{A}_I(D)$. The description of all of the maps on $V(D)$ and the proof of [12, Proposition 5.2] depend only on the form of their Heegaard diagram near crossings, so they apply unchanged to our master diagram. Taken together, the maps defined by counting appropriate holomorphic disks near crossings in the master diagram form an endomorphism of $V(D)$. Lemma 4.6 of [12] shows that $V(D)$ with this endomorphism is quasi-isomorphic to the chain complex $\underline{CFK}^-(D)$, which is the twisted knot Floer homology of the classical knot $D$, defined via the traditional holomorphic disks construction and regarded as an $\mathbb{F}_2[x_0][[t]]$-module. Again, the arguments depend only on the properties of the master diagram near crossings in $D$, so they carry through unchanged to our situation. Therefore, as in [12, Theorem 4.4], the filtration on $V(D)$ gives rise to a spectral sequence with $E_1$ page

$$\bigoplus_{I \in \{0,1\}^m} H_*\big(\underline{CFK}^-(D_I) \otimes M_I\big)\,,$$

with $d_1$ differential the zip and unzip maps defined algebraically, and converging to $\underline{HFK}^-(D)$.

Finally, in Section 5, Ozsváth and Szabó argue that this spectral sequence collapses after the $E_1$ stage for grading reasons. The gradings in this paper are defined identically to those in [12], so the same argument shows that the spectral sequence here collapses. The immediate result is an isomorphism of $\mathbb{F}_2[x_0][[t]]$-modules

$$H_*\left(\bigoplus_{I\in\{0,1\}^m} H_*\big(\underline{CFK}^-(D_I)\big)\otimes M_I\right)\cong H_*\big(\underline{CFK}^-(D)\big)$$

Inverting $t$ in the ground ring throughout the spectral sequence, then applying the isomorphism from (21) allows us to identify the left side with the cube of resolutions complex $C(D)$ used in this paper:

$$H_*\left(C(D)\otimes_{\mathcal{R}[\underline{x}(D)]}\widehat{\mathcal{R}[\underline{x}(D)]}\otimes\mathbb{F}_2\right)\cong H_*\big(\underline{CFK}^-(D)\otimes_{\mathbb{F}_2[[t]]}\mathbb{F}_2[t^{-1},t]]\big)$$

A standard theorem about twisted coefficients in knot Floer homology, stated as [12, Lemma 2.2], completes the identification with $H_*\left(CFK^-(D)\right)\otimes_{\mathbb{F}_2}\mathbb{F}_2[t^{-1},t]]$. The reduced statement follows similarly. □

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